Locally Adapted Resolutions for Potential Field Modelling and Inversion

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Introduction

For the analysis and inversion of potential field data, several approaches are known:

- expansion in orthonormal polynomials,
- spline approximation,
- wavelet analysis.

Each method uses particular basis systems. Whereas the classical first approach uses global basis functions (such as spherical harmonics), the latter two proved to be advantageous for high-resolution analyses due to the construction of localized basis functions (spline basis functions, scaling functions, and wavelets). However, the drawback of the wavelets developed for such applications is their inflexibility with respect to heterogeneous data. Furthermore, the numerical limit of spline methods is given by the size of the dense matrix that has to be inverted in the algorithm. Moreover, all methods are incapable of using a mixture of different kinds of basis functions.
We will show here that an adaptation of a greedy algorithm developed for the Euclidean setting (see e.g. [4, 6]) allows us to overcome these drawbacks. We iteratively increase the resolution and accuracy of the obtained model, which avoids the inversion of a matrix and allows theoretically an unlimited number of summands in the expansion.
Moreover, the expansion functions are chosen from a so-called dictionary, which may be a very heterogeneous mixture of all kinds of trial functions.

The idea

We intend to construct an approximate solution of the form

$$F = \sum_k \alpha_k d_k$$

in the following way:

- Each summand is chosen iteratively, i.e. we move from $F_n := \sum_{k=1}^{n} \alpha_k d_k$ to $F_{n+1} := \sum_{k=1}^{n+1} \alpha_k d_k$.
- Every d_k is a function which is selected from a dictionary $\mathcal{D} \subset \mathrm{L}^2(\mathcal{B})$, which is a redundant system of functions, i.e. it contains global functions $G^{\mathrm{I}}_{m,n,j}$ and kernels $K^{\mathrm{I}}_{h_j}(x_{i,j}, \cdot)$ with different localizations (controlled by h_j) and different centers $x_{i,j}$.
- The coefficients $\alpha_k \in \mathbb{R}$ are chosen in combination with the dictionary elements d_k.
- The objective is to minimize

$$\left\| y - \mathcal{F}\left(F_n + \alpha_{n+1} d_{n+1}\right) \right\|^2_{\mathbb{R}^l} + \lambda \left\| F_n + \alpha_{n+1} d_{n+1} \right\|^2_{\mathrm{L}^2(\mathcal{B})}$$

for given $\mathcal{F} : \mathrm{L}^2(\mathcal{B}) \to \mathbb{R}^l$ (linear) and $y \in \mathbb{R}^l$, where λ is the regularization parameter.

The algorithm RFMP

1) Start with $F_0 := 0$, $R^0 := y$, $n := 0$.
2) Build $F_{n+1} := F_n + \alpha_{n+1} d_{n+1}$ such that

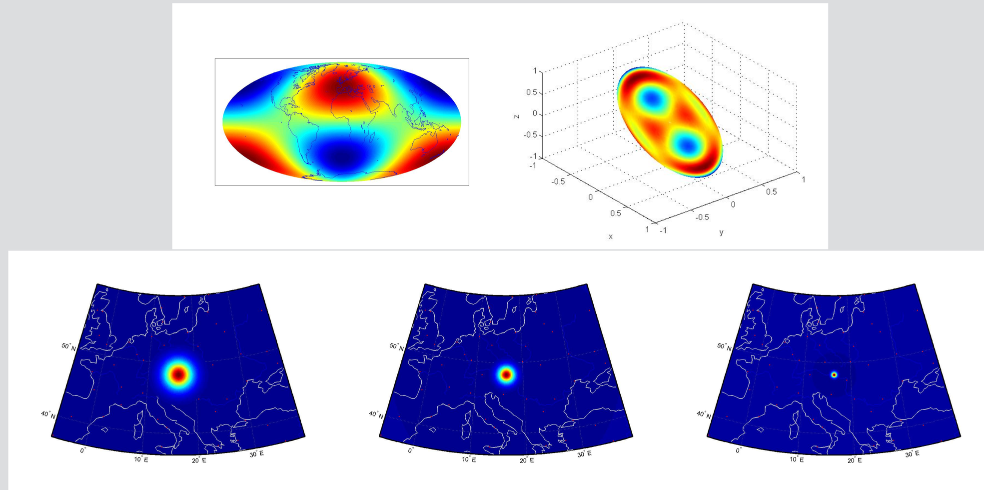
$$d_{n+1} = \operatorname*{argmax}_{d \in \mathcal{D}} \left| \frac{\langle R^n, \mathcal{F}d \rangle_{\mathbb{R}^l} - \lambda \langle F_n, d \rangle_{\mathrm{L}^2(\mathcal{B})}}{\sqrt{\|\mathcal{F}d\|^2_{\mathbb{R}^l} + \lambda \|d\|^2_{\mathrm{L}^2(\mathcal{B})}}} \right| \quad \text{and}$$

$$\alpha_{n+1} := \frac{\langle R^n, \mathcal{F}d_{n+1} \rangle_{\mathbb{R}^l} - \lambda \langle F_n, d_{n+1} \rangle_{\mathrm{L}^2(\mathcal{B})}}{\|\mathcal{F}d_{n+1}\|^2_{\mathbb{R}^l} + \lambda \|d_{n+1}\|^2_{\mathrm{L}^2(\mathcal{B})}}$$

3) Update the residual $R^{n+1} := R^n - \mathcal{F}(\alpha_{n+1} d_{n+1})$.
4) Stop or increase n by 1 and go to Step 2.

The dictionary

Examples of basis functions for the 3d-ball, plotted on the surface or a planar cut

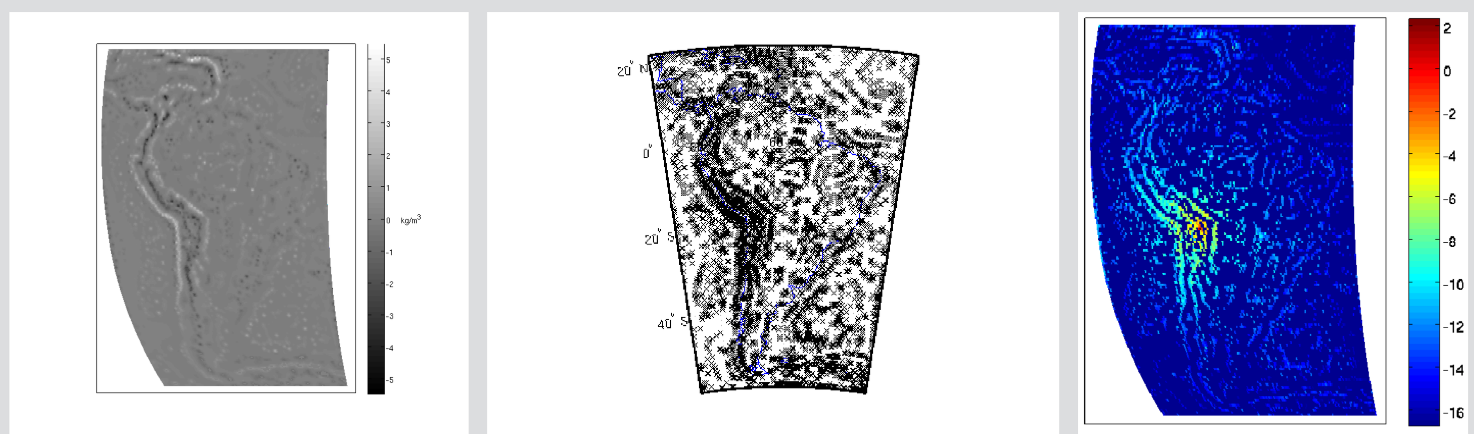


Application: Inversion of gravitational data

- 25,440 (simulated) data of the EGM2008 potential on a point grid 7 km above the Earth
- dictionary of the form described above
- the regularization parameter $\lambda = 4.6416$ is chosen via the L-curve method
- the algorithm is truncated after 20,000 summands were chosen for the expansion

Numerical result

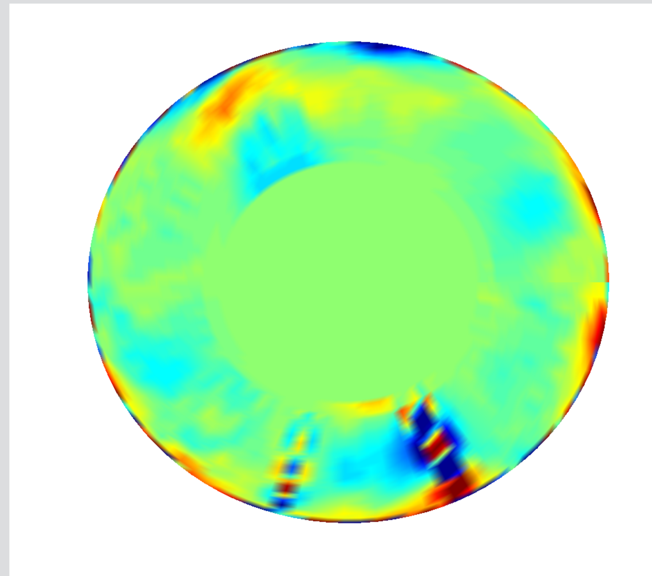
Harmonic density variations (left-hand), chosen centres of the localized basis functions (middle), and local influence of the basis functions (right-hand)



result | centres | local influence

Combined inversion (gravitation and normal modes)

We combine 1,560 gravitational data points (from EGM2008) and 1,738 splitting functions coefficients (data courtesy of Arwen Deuss, Cambridge) and stop the algorithm at $n = 10{,}000$.



equatorial cut (with enlarged upper 300 km)

Conclusions

The new algorithm has several new features in comparison to previous approaches:

- The obtained solution is sparse, i.e. the same number of expansion functions yields a more accurate result.
- Different kinds of basis functions can be combined. The algorithm automatically chooses those which yield the best possible reduction of the approximation error.
- There is no numerical limit any more for the number of used basis functions.
- Different data types may be mixed, where much more data than previously may be used.

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References

- P. Berkel, D. Fischer, V. Michel: Spline Multiresolution and Numerical Results for Joint Gravitation and Normal-mode Inversion with an Outlook on Sparse Regularization, *Int. J. Geomath.*, 1:167-204, 2011.
- D. Fischer: Sparse Regularization of a Joint Inversion of Gravitational Data and Normal Mode Anomalies, PhD thesis, Geomathematics Group, University of Siegen, submitted, 2011.
- D. Fischer, V. Michel: Sparse Regularization of Inverse Gravimetry — Case Study: Spatial and Temporal Mass Variations in South America, Preprint, 2011.
- S.G. Mallat, Z. Zhang: Matching Pursuits with Time-Frequency Dictionaries, *IEEE Trans. Signal Process.*, 41:3397-3415, 1993.
- V. Michel: Tomography — Problems and Multiscale Solutions, in: "Handbook of Geomathematics" (W. Freeden, M.Z. Nashed, T. Sonar, eds.), 2010, pp. 949-972.
- P. Vincent, Y. Bengio: Kernel Matching Pursuit, *Machine Learning*, 48:169-191, 2002.