Physical vs. Numerical Dispersion
in
Nonhydrostatic Ocean Modeling

EGU April 2011 – OS4.4 EGU2011-385

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“ … focused examination and testing of ocean model resolution must be a prime goal of ocean climate model development over the next decades. ”
- **Griffies [2000]**

“ … models are imperfect tools. Furthermore, discussion and comprehension of the results from complex models depend on the results from idealized models.”
- **Philander [2009]**

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Abstract

Many popular ocean models second-order accurate, inducing numerical dispersion generated from odd-order terms in the truncation error.

Internal waves are (often) a dynamical balance between nonlinearity and nonhydrostasy (physical dispersion).

Numerical dispersion mimics physical dispersion due to nonhydrostasy.

To lowest order, the ratio of numerical to physical dispersion is

$$\boxed{\Gamma = K\lambda^2}$$

K is typically an $O(1)$ constant

$\lambda \equiv \frac{\Delta x}{h_1}$ is the grid leptic ratio, or lepticity

Δx is the horizontal grid spacing

h_1 is the upper layer (pycnocline) depth

We derive this relationship for simple models (KdV equation), and show that it holds in a real ocean model (SUNTANS – Fringer et. al. 2006).

To ensure relative dominance of physical over numerical dispersive effects:

$$\boxed{\Gamma << 1} \longleftrightarrow \boxed{\lambda \approx O(0.1)} \longleftrightarrow \boxed{\Delta x < h_1}$$

Ocean Modeling: Range of Scales

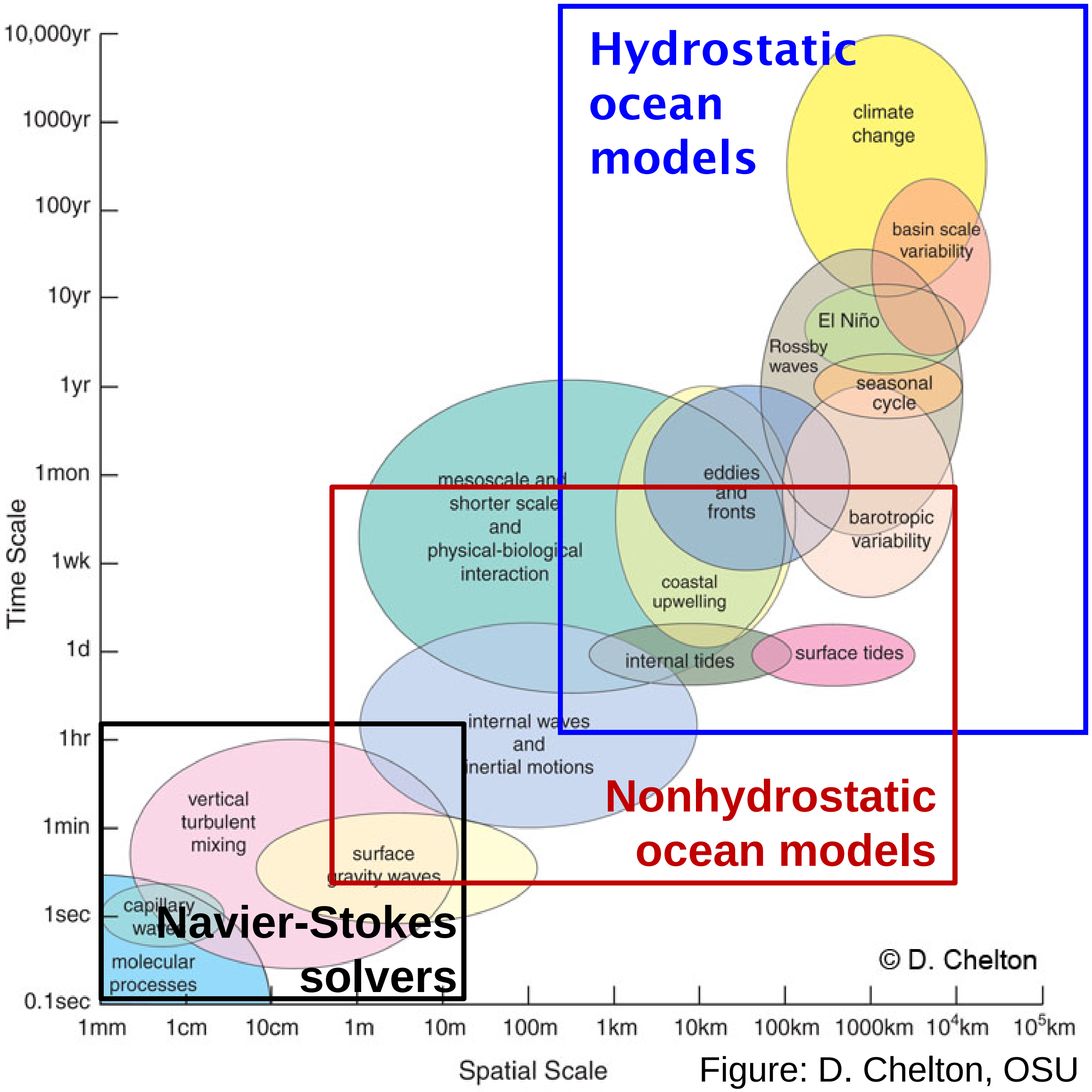


Figure: D. Chelton, OSU

Most ocean models are traditionally hydrostatic.

The hydrostatic approximation is valid when

Horizontal scale >> Vertical scale

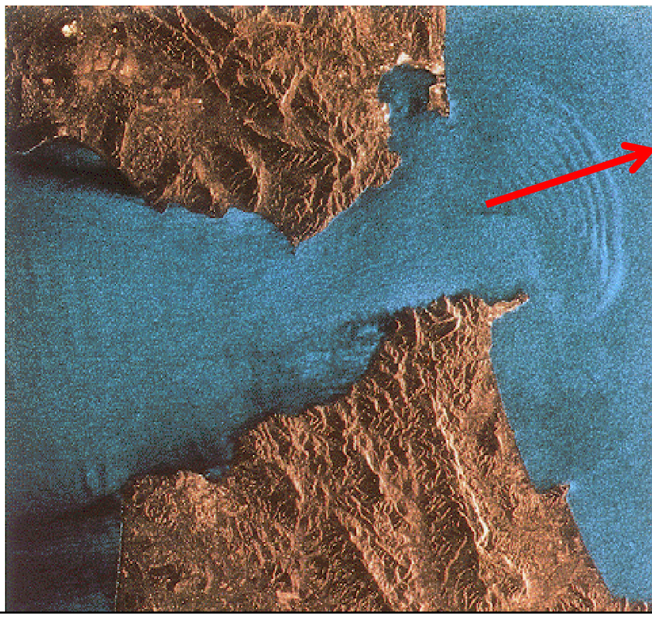
Nonhydrostatic models compute motions for which

Horizontal scale ~ Vertical scale

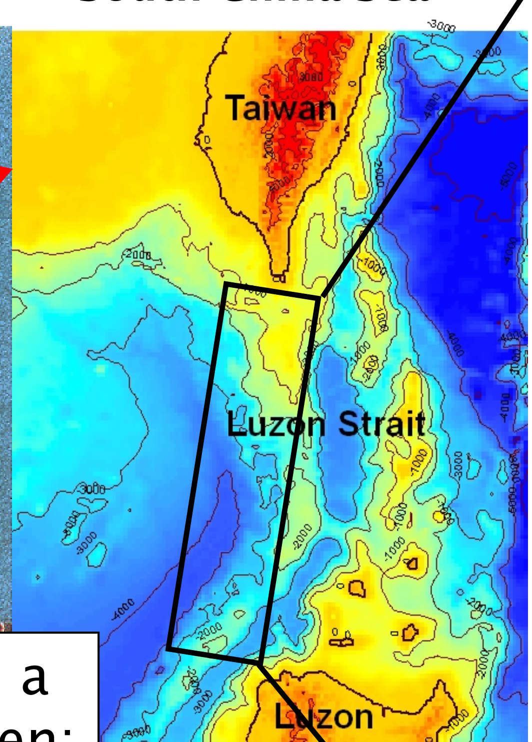
This requires computationally expensive solution of a 3D elliptic equation.

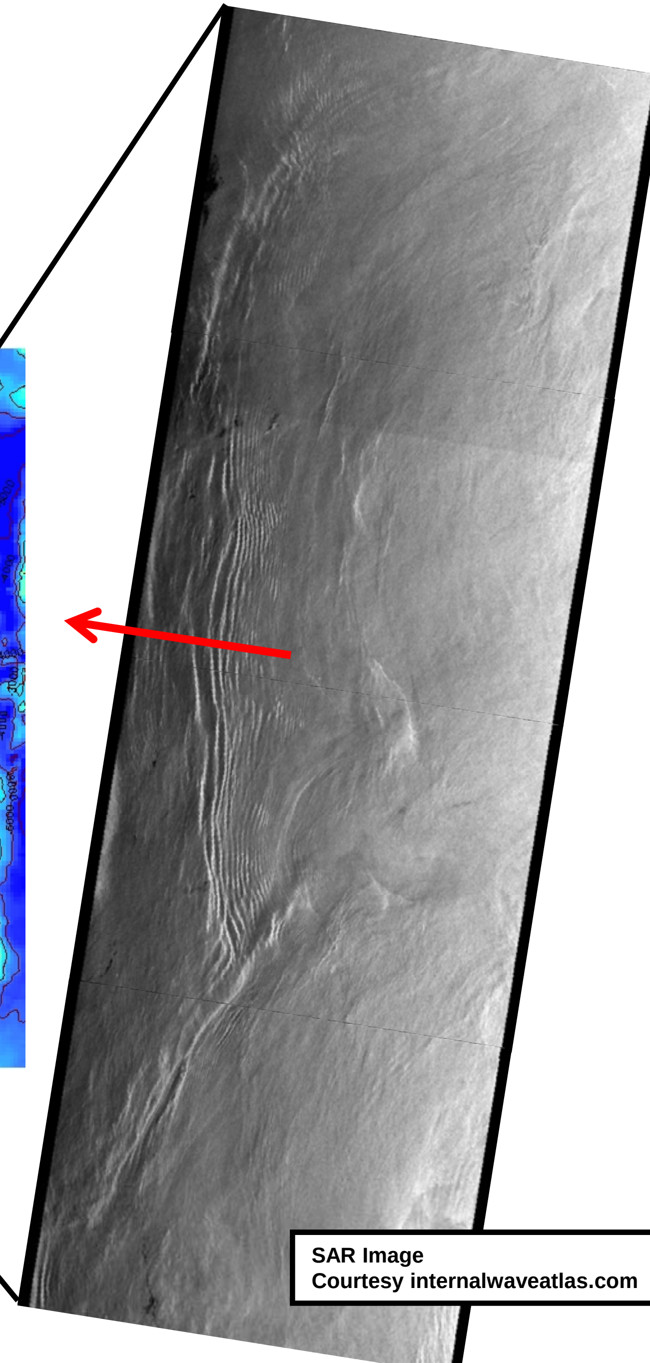
Internal Waves

Straight of Gibraltar



South China Sea





SAR Image
Courtesy internalwaveatlas.com

Internal waves are (often) a dynamical balance between:

Nonlinearity **&** **Nonhydrostasy (physical dispersion)**

$$\delta = \frac{a}{H} = \frac{\text{wave amplitude}}{\text{depth}}$$

$$\varepsilon = \frac{H}{L} = \text{aspect ratio}$$

The KdV equation

When modeling solitary waves, the behavior of a fully nonhydrostatic ocean model can be well approximated with the KdV (Korteweg and de-Vries, 1895) equation:

Ocean Model:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -g\nabla\eta - g\nabla r - \nabla q$$

Requires solution of 3D elliptic equation

Unsteadiness

Nonlinear advection of momentum

Linear baroclinic pressure gradient

Nonhydrostatic pressure gradient

KdV:

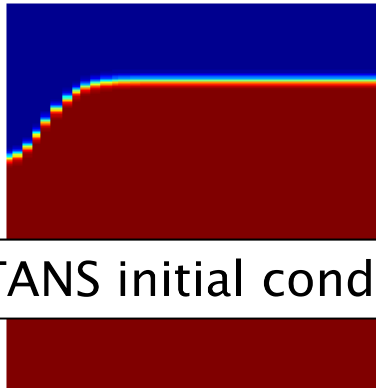
$$\frac{\partial \xi}{\partial t} - \frac{3}{2}\delta\xi\frac{\partial \xi}{\partial x} = -\frac{\partial \xi}{\partial x} - \frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3} + O\left(\delta^2, \varepsilon^4, \delta\varepsilon^2\right)$$

The well-known solution to the KdV equation is:

$$\xi(x,t) = -a\,\mathrm{sech}^2\left(\frac{x-ct}{L_0}\right)$$

$$L_0 = \sqrt{\frac{4}{3a}\frac{\varepsilon^2}{\delta}} = \sqrt{\frac{4}{3a}\frac{\text{dispersion}}{\text{nonlinearity}}}$$

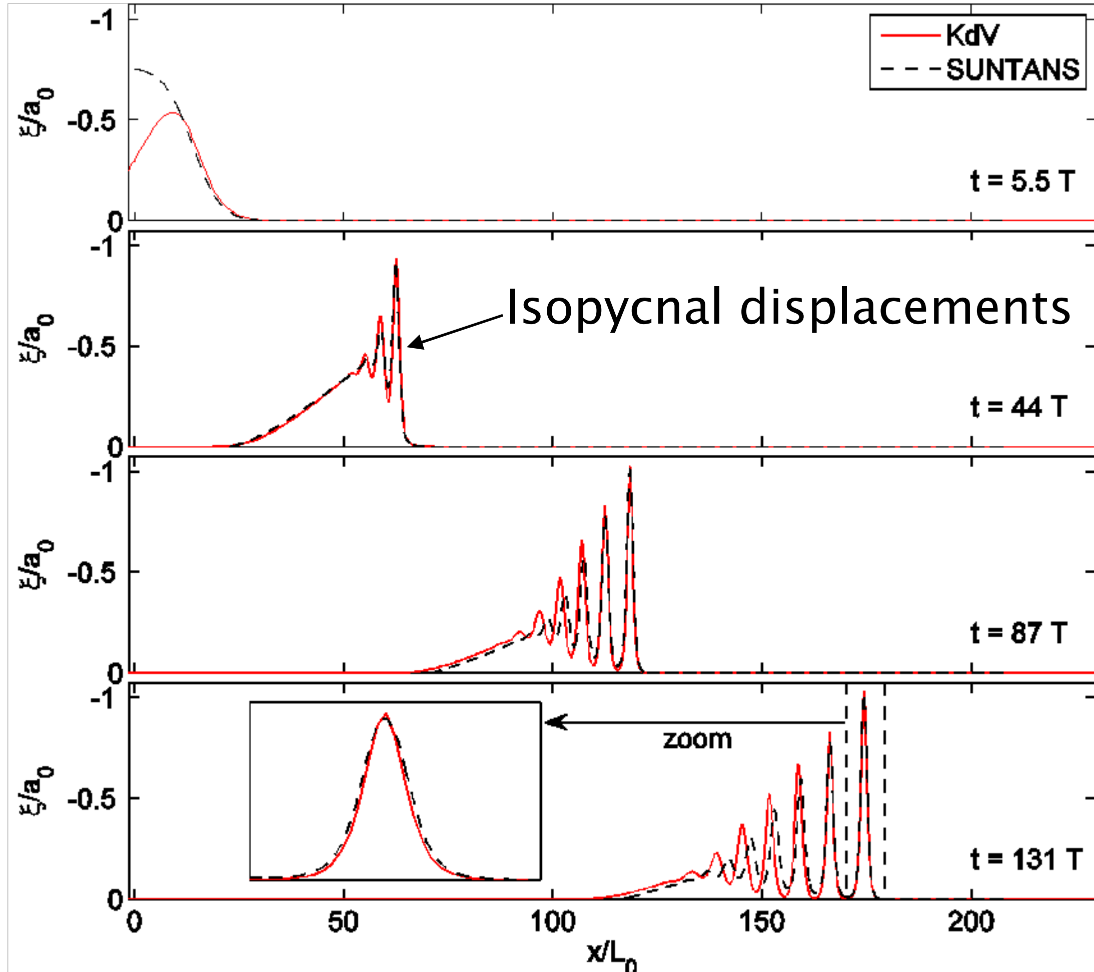
Comparison of KdV to the nonhydrostatic model SUNTANS



SUNTANS initial condition

KdV model uses coefficients that account for continuous stratification (Liu et al. 2004).

Since SUNTANS is well-approximated by KdV, we can analyze the numerical properties of the KdV equation to quantify the dispersive error in more complex models.



Numerical Discretization of KdV Eq.

・SUNTANS and many ocean models discretize the equations with second-order accuracy in time and space.
(e.g. POM, Blumberg and Mellor, 1987; MICOM, Bleck et al., 1992; MOM, Pacanowski and Griffes, 1999).

・A second-order accurate discretization of the KdV equation using "leap-frog" (i.e. POM) is given by

$$\frac{\partial \xi}{\partial t} + \left(1 - \frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x} + \frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3} = 0$$

$$\frac{\xi_i^{n+1} - \xi_i^{n-1}}{2\Delta t} + \left(1 - \frac{3}{2}\delta\xi_i^n\right)\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} + \frac{\varepsilon^2}{6}\frac{\frac{1}{2}\xi_{i+2}^n - \xi_{i+1}^n + \xi_{i-1}^n - \frac{1}{2}\xi_{i-2}^n}{\Delta x^3} = 0$$

・Taylor series expansions can be used to determine the truncation error, or modified equivalent PDE:

$$\frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta x} = \left.\frac{\partial \xi}{\partial x}\right|_i^n + \frac{\Delta x^2}{6}\left.\frac{\partial^3 \xi}{\partial x^3}\right|_i^n + \frac{\Delta x^4}{120}\left.\frac{\partial^5 \xi}{\partial x^5}\right|_i^n + \frac{\Delta x^6}{5040}\left.\frac{\partial^7 \xi}{\partial x^7}\right|_i^n + O\left(\Delta x^8\right)$$

Modified equivalent KdV equation

The discrete KdV equation produces a modified equivalent PDE (Hirt 1968) which introduces new terms due to discretization errors:

KdV:

$$\frac{\partial \xi}{\partial t}+\left(1-\frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x}+\frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3}=0$$

Modified KdV:

$$\frac{\partial \xi}{\partial t}+\left(1-\frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x}+\left(1+\Gamma\right)\frac{\varepsilon^2}{6}\frac{\partial^3 \xi}{\partial x^3}=O\left(\Delta x^4,\Delta t^4,\delta\Delta x^2,\varepsilon^2\Delta x^2\right)$$

The numerical discretization of the first-order derivative produces numerical dispersion. Errors in the nonlinear term are smaller by ~ a factor δ.

$$\Gamma=\frac{\text{numerical dispersion}}{\text{physical dispersion}}=K\left(\frac{\Delta x'}{\varepsilon}\right)^2=K\left(\frac{\Delta x}{h_1}\right)^2=K\lambda^2$$

K is typically an $O(1)$ constant

$\lambda\equiv\frac{\Delta x}{h_1}$ is the grid leptic ratio, or lepticity

Scotti & Mitran (2008)

Δx is the horizontal grid spacing

h_1 is the upper layer (pycnocline) depth

Numerical dispersion in hydrostatic and nonhydrostatic ocean modeling

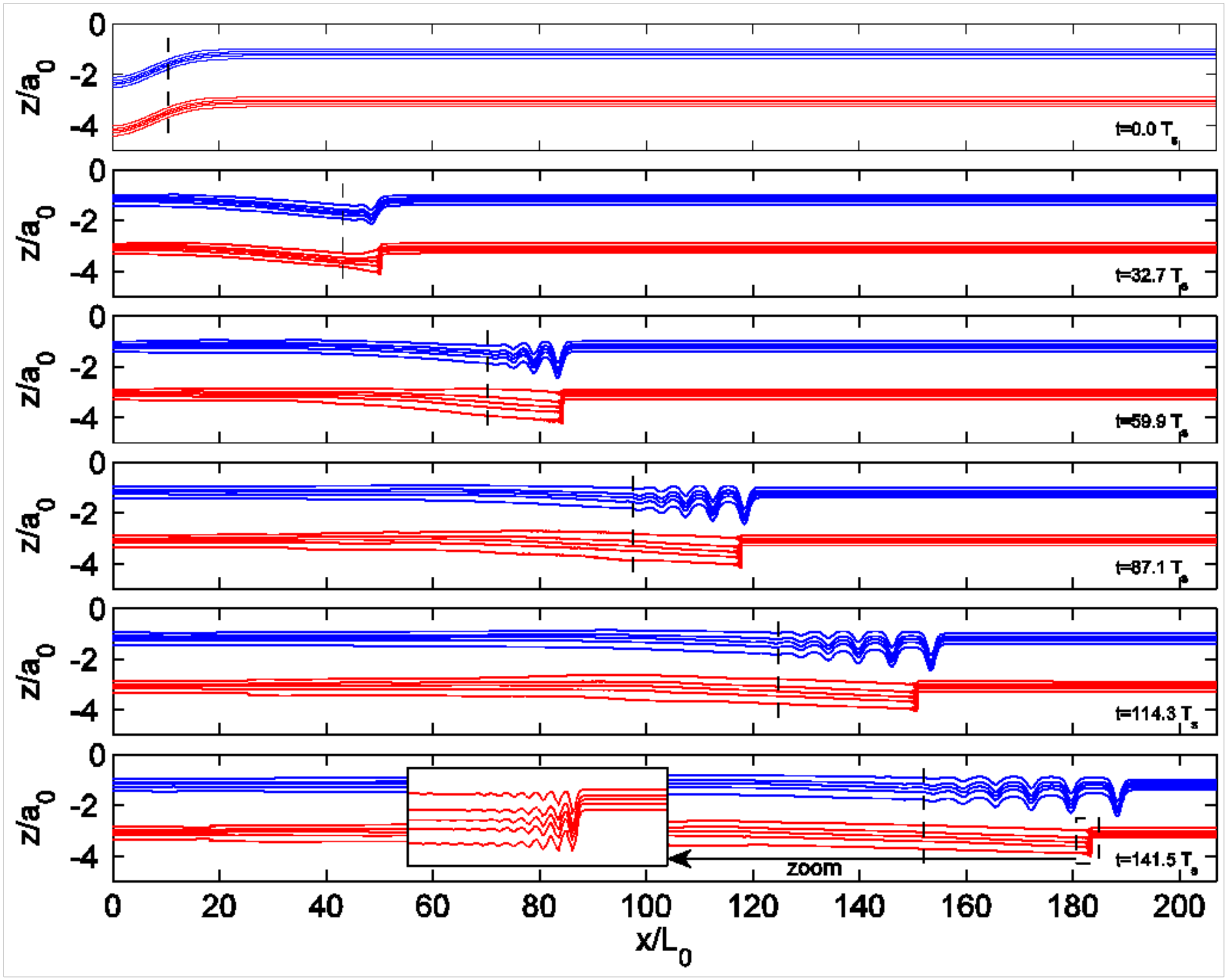
- "Nonhydrostatic" models possess both physical & numerical dispersion:

$$\frac{\partial \xi}{\partial t}+\left(1-\frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x}+\left(1+\Gamma\right)\frac{\varepsilon^{2}}{6}\frac{\partial^{3}\xi}{\partial x^{3}}=0$$

- "Hydrostatic" models possess only numerical dispersion:

$$\frac{\partial \xi}{\partial t}+\left(1-\frac{3}{2}\delta\xi\right)\frac{\partial \xi}{\partial x}+\Gamma\frac{\varepsilon^{2}}{6}\frac{\partial^{3}\xi}{\partial x^{3}}=0$$

Hydrostatic vs. Nonhydrostatic Ocean Model (SUNTANS)



$$\Delta x = h_1 / 4$$

Hydrostatic model dispersion (numerical):

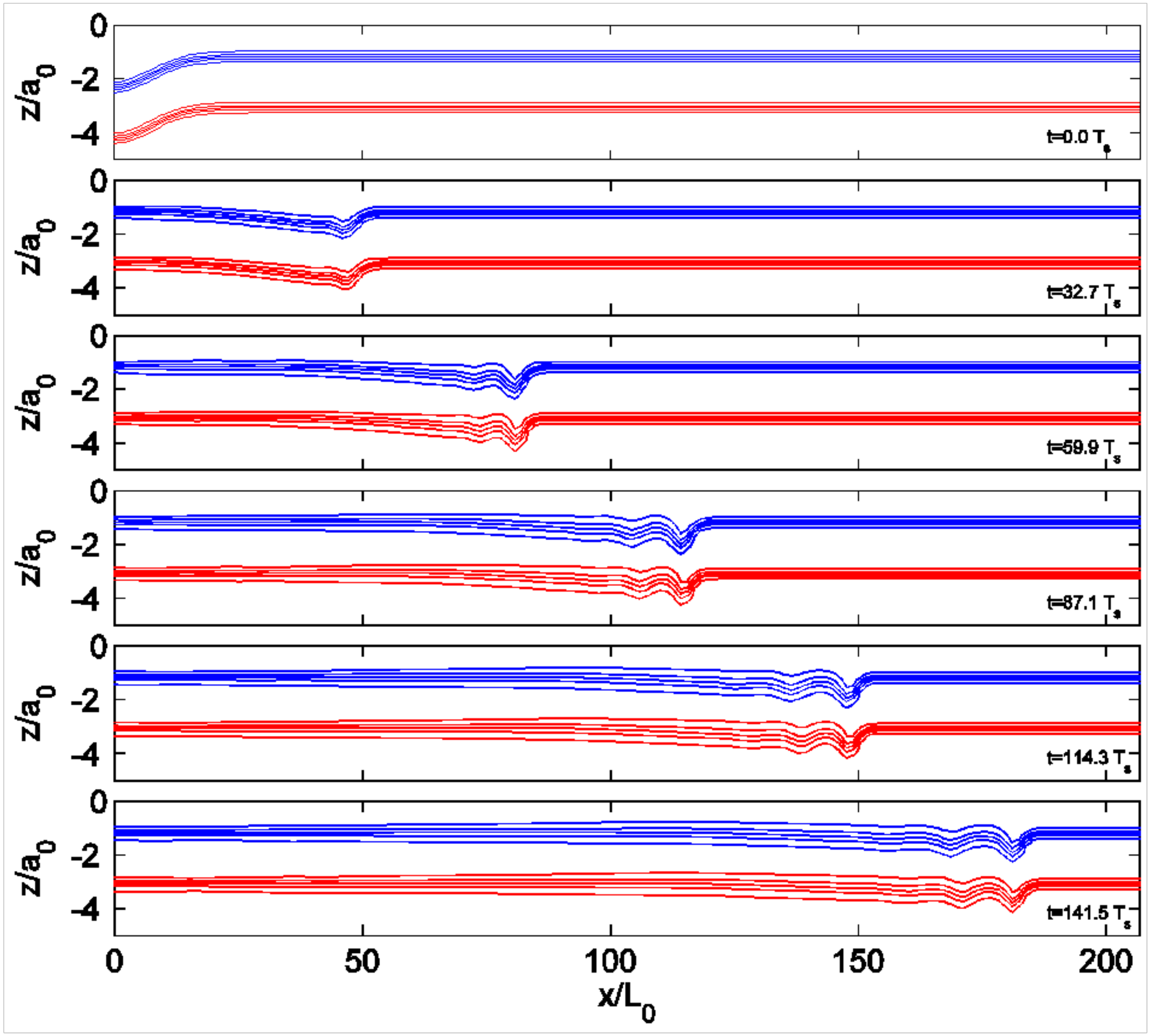
$$\Gamma = 0.005$$

Nonhydrostatic model dispersion (physical+numerical):

$$1 + \Gamma = 1.005$$

Numerical dispersion is **200** times smaller than physical dispersion.

Hydrostatic vs. Nonhydrostatic Ocean Model (SUNTANS)



$$\Delta x = 8h_1$$

Hydrostatic model dispersion (numerical):

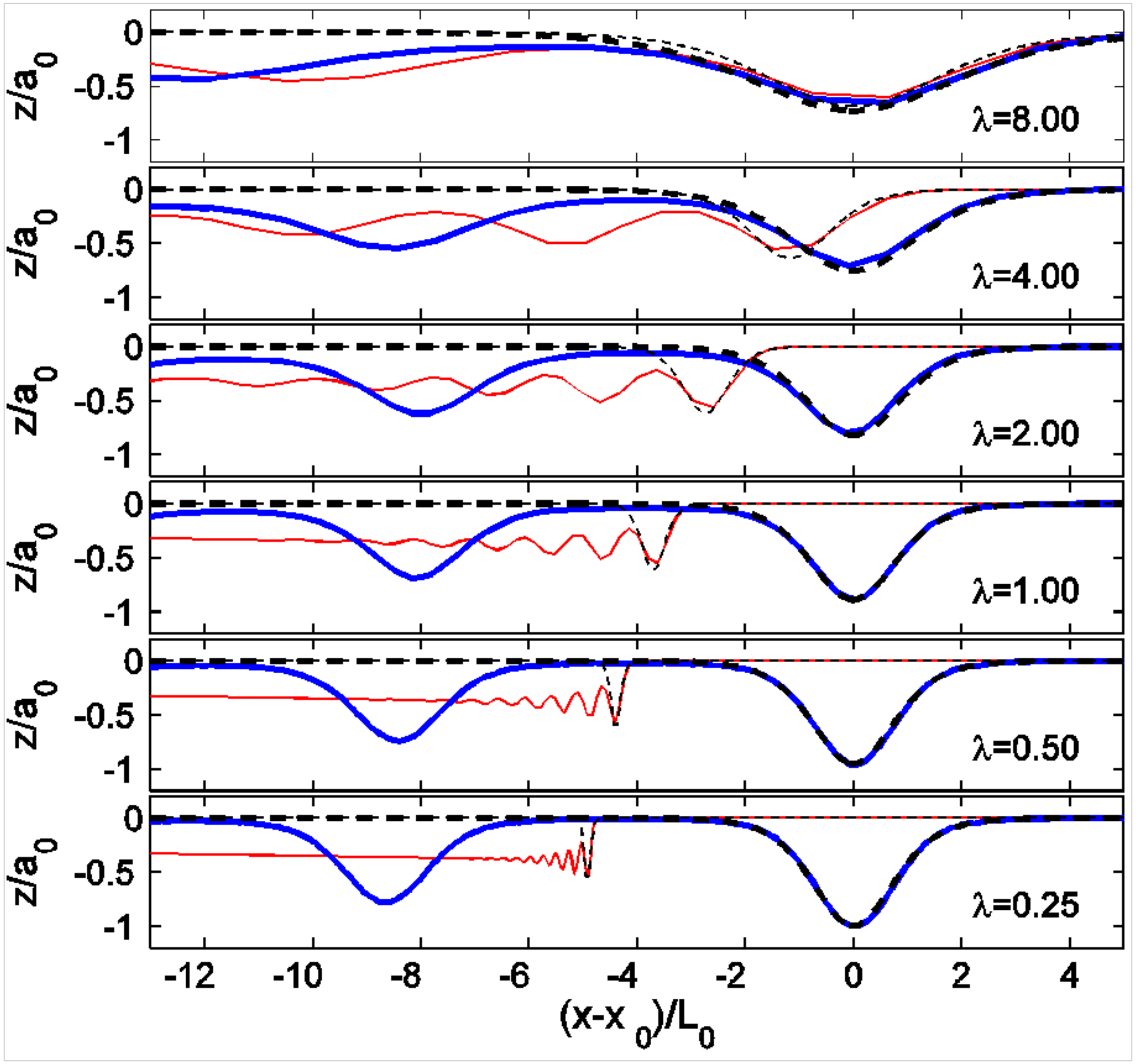
$$\Gamma = 5$$

Nonhydrostatic model dispersion (physical+numerical):

$$1+\Gamma = 6$$

Numerical dispersion is **5** times larger than physical dispersion.

Effects of $\lambda = \Delta x / h_1$ (grid resolution)



Hydrostatic vs. Nonhydrostatic SUNTANS

Hydrostatic and nonhydrostatic models produce the same "numerical solitary-like waves" for large λ.

Small λ

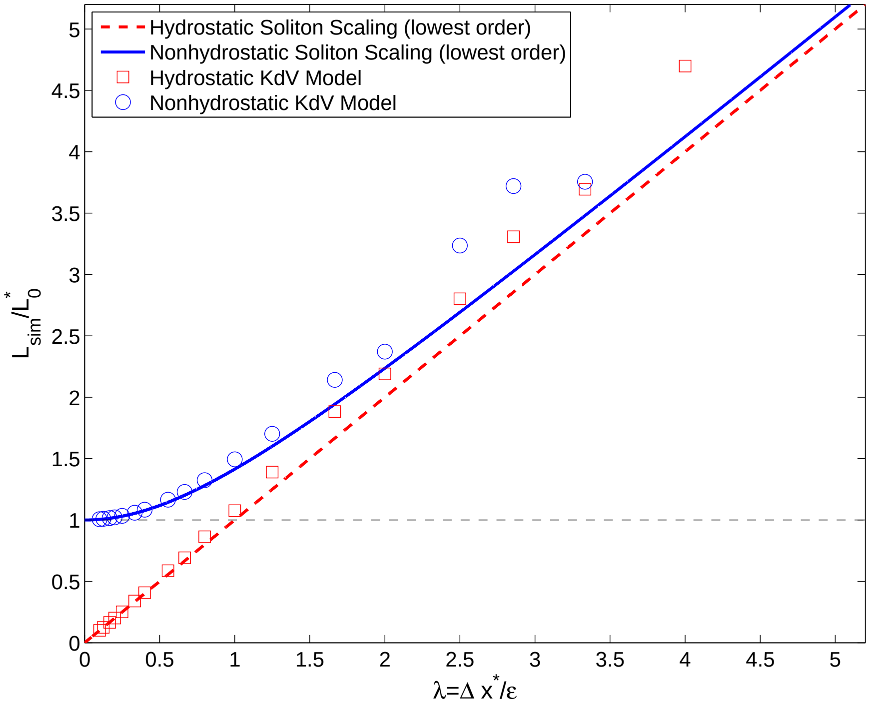
Large λ

Hydrostatic models produce sharp fronts due to small numerical dispersion.

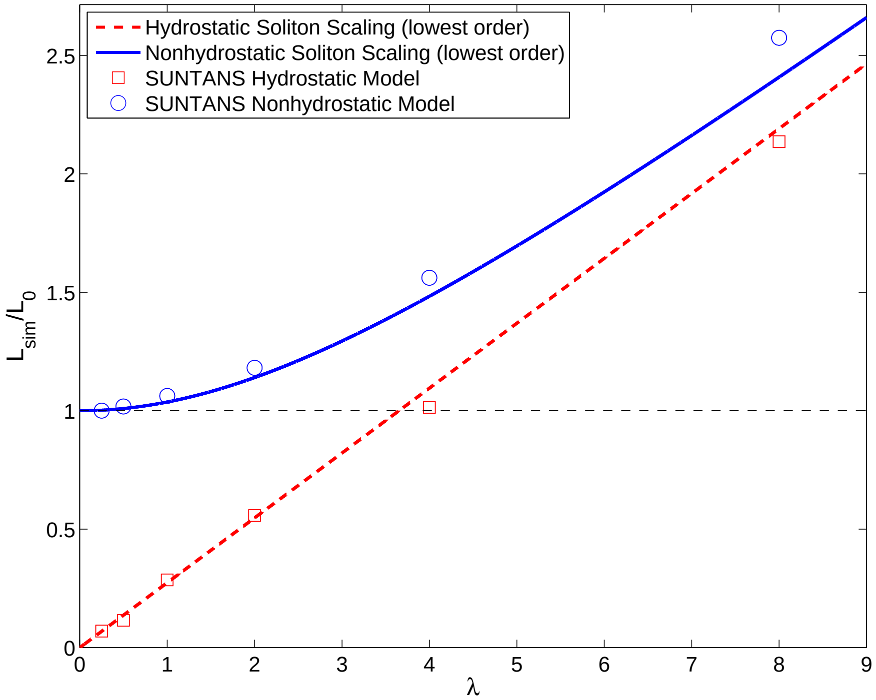
Nonhydrostatic models converge to the correct solitary wave for small λ.

Hydrostatic vs. Nonhydrostatic Modeled soliton widths

KdV equation



Ocean Model (SUNTANS)



$$L / L_0 = \sqrt{1+\Gamma}$$

$$L_h / L_0 = \sqrt{\Gamma}$$

$$\boxed{\Gamma = K\lambda^2}$$

$$L / L_0 = \sqrt{1+K\lambda^2}$$

$$L_h / L_0 = \lambda\sqrt{K}$$

Conclusions

- To resolve nonhydrostatic effects in internal gravity waves, the grid lepticity must satisfy $\lambda = \Delta x / h_1 \approx O(0.1)$.
- Large λ leads to excessive numerical dispersion and hydrostatic and nonhydrostatic models produce the same (incorrect) results.
- This analysis assumes second-order accuracy. Third-order accurate models (in both time and space) would not produce (lowest order) numerical dispersion and provide more accurate results.
- This condition $\Delta x < h_1$ may be a significant additional resolution requirement beyond the current-state-of-the art in ocean modeling of internal waves.

Acknowledgements

- **US Department of Energy (DOE)**

  Computational Science Graduate Fellowship (CSGF)

- **Office of Naval Research (ONR)**

  Under grants N00014-10-1-0521 and N00014-08-1-0904

- **EGU organizing committee** for travel support