This poster participates in OSP

Outstanding Student Poster Contest

Computing streamfunction and velocity potential in a limited domain of arbitrary shape

Jie CAO IAP, Chinese Academy of Sciences, Beijing, CHINA; CIMMS, University of Oklahoma, USA

Qin XU NOAA/National Severe Storms Laboratory, Norman, USA

1. Introduction

Streamfunction and velocity potential, ψ and χ, are useful scalars of flow fields and powerful tools for studies of fluid dynamics. They provide useful tools for weather and climate map analyses, for deriving velocity error covariance representations in large-scale data assimilation.

However the direct usage of ψ and χ in the basic dynamics of the rotational and divergent components of the global or regional atmospheric circulation has not been well studied. Main reason lies on the lack of ways to compute ψ and χ precisely and conveniently.

Problems to be solved: $\Delta\chi = \alpha$, $\Delta\psi = \zeta$, inside the domain;

$\partial_n\chi - \partial_s\psi = v_n$, $\partial_s\chi + \partial_n\psi = v_s$, along the boundary.

Objective: to explore the utilities of classic integral formulae for computing ψ and χ, and to design efficient numerical methods to **accurately compute ψ and χ in both regular and irregular domains either in the presence or absence of data hole.**

2. Theory & Integral formulae

Method: to minimize the difference between the domain-integrated kinetic energy of the original horizontal velocity and that of the reconstructed one, i.e., $J = \int_D d\mathbf{x}\,|\nabla\chi + \mathbf{k}\times\nabla\psi - \mathbf{v}|^2$.

$$(\chi,\ \psi) = (\chi_i,\ \psi_i) + (\chi_e,\ \psi_e) + (\chi_d,\ \psi_d)$$

➢The internally induced solution

$\chi_i = (2\pi)^{-1}\int_D dx'\,\alpha(x')\ln|x-x'|$; $\psi_i = (2\pi)^{-1}\int_D dx'\,\zeta(x')\ln|x-x'|$.

(1) is solved in unbounded domain, therefore, no B.C.

(2) satisfies: $\Delta\chi_i = \alpha$, and $\Delta\psi_i = \zeta$.

➢The data-hole induced solution

$\chi_d = \sum_1(2\pi)^{-1}\ln|x-x_k|\int_{Sk}ds(v_n-v_{ni})$; $\psi_d = \sum_1(2\pi)^{-1}\ln|x-x_k|\int_{Sk}ds(v_{si}-v_s)$.

(1) is constructed by placing point sources of α and ζ inside each data hole to represent the net effects of the integrated flux and circulation around the data hole.

(2) should be subtracted from the total solution to modify B.C. and solvability conditions for $(\chi_e,\ \psi_e)$.

➢The externally induced solution

Approach 1: $(\chi_e,\ \psi_e) = (0, -Im\,\omega_e)$, where the complex velocity potential $\omega_e = \varphi_e - i\psi_e = (i2\pi)^{-1}\int_S dz'\,\omega_e(z')/(z'-z)$, $\omega_e(s) = \omega_e(s_0) + \int_0^s ds'[(v_s - v_{si}) + i(v_n - v_{ni})]$ on the boundary S, and $\partial_n\psi_e = \partial_s\varphi_e$ along S.

Approach 2:

2.1 With Dirichlet B.C. :

$\psi_e(\mathbf{x}) = \int_S ds'\,\mu(\mathbf{x}')\mathbf{n}'\bullet\mathbf{r}/r^2$ for $\mathbf{x}$ in D, and the density function $\mu(\mathbf{x}')$ satisfies:

$\psi_e(\mathbf{x}) = \pi\mu(\mathbf{x}) + \int_S ds'\,\mu(\mathbf{x}')\mathbf{n}'\bullet\mathbf{r}/r^2$ for $\mathbf{x}$ on S.

2.2 With Neumann B.C. :

$\psi_e(\mathbf{x}) = -\int_S ds'\,\nu(\mathbf{x}')\ln r$ for $\mathbf{x}$ over D, and the density function $\nu(\mathbf{x}')$ satisfies:

$\partial_n\psi_e(\mathbf{x}) = \pi\nu(\mathbf{x}) + \int_S ds'\,\nu(\mathbf{x}')\mathbf{n}\bullet\mathbf{r}/r^2$ for $\mathbf{x}$ on S.

2.3 With tangential-derivative B.C. :

$\psi_e(\mathbf{x}) = -\int_S ds'\,\nu(\mathbf{x}')\ln r$ for $\mathbf{x}$ over D, and $\nu(\mathbf{x}')$ satisfies:

$-\int_S ds'\,\nu(\mathbf{x}')\mathbf{s}\bullet\mathbf{r}/r^2 = v_{ni} - v_n$ for $\mathbf{x}$ on S.

(1) This part contains the null space which leads to non-unique solution.

(2) χ_e can be solved exactly the same as ψ_e.

Cons and Pros when compare to other methods:

√ The non-uniqueness of solution and compatibility between the coupled B.C. are well taken care of.

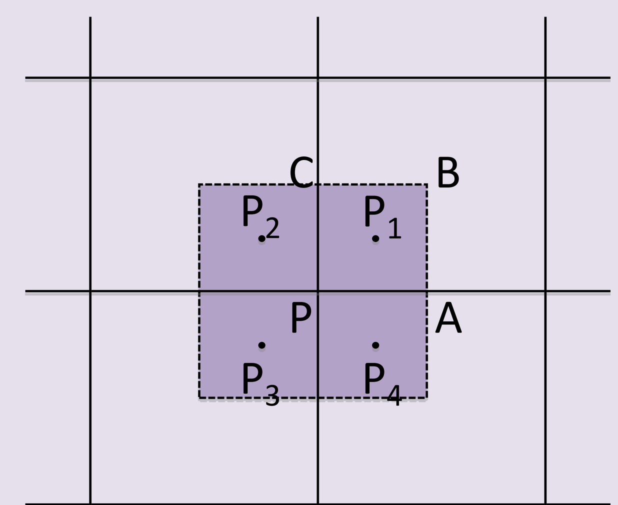
√ This method can solve for limited domains of arbitrary shape with inner data holes.

? How to design numerical schemes to reduce discretization error?

? How to design an automated algorithm to format the B.C. for the integrations along irregular external and internal boundaries?

3. Numerical methods & Exps.

➢ Discretization schemes for $(\chi_i,\ \psi_i)$: to deal with the singular point P.



S1: the solution and source fields are of the same resolution; but staggered by ½ $\triangle d$.

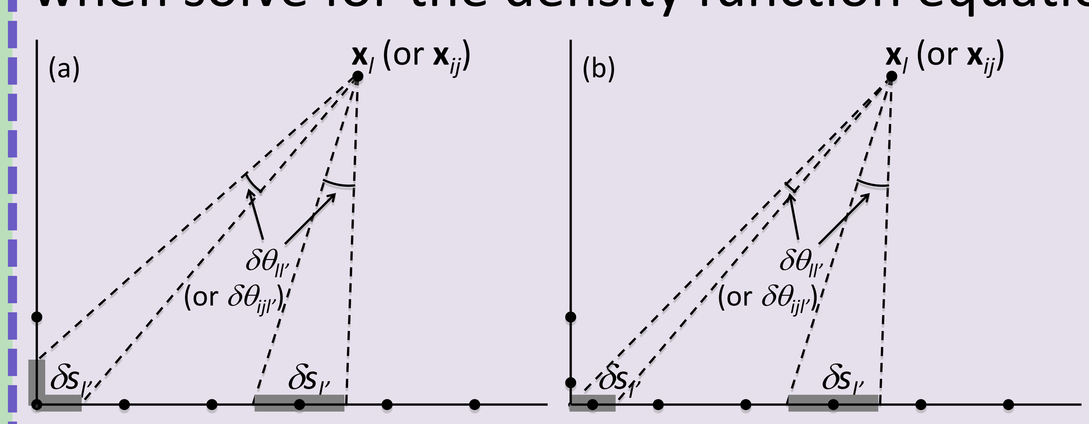
S2: the same grids; but the grid is nested locally around P into $P_{1,2,3,4}$.

S3: similar to S2, except that the nested discrete integrations around P are replaced by piecewise continuous integration.

Exp 1: $\psi_i^{true} = \sum_m\sum_n \cos(m\pi x/L)\sin(n\pi y/L)$.

Scheme	S1	S2	S3
SCC	1.000	1.000	1.000
RRD	0.888%	0.856%	0.721%

➢ Discretization schemes for $(\chi_e,\ \psi_e)$: to reduce the error near corners when solve for the density function equation.



SD1: Dirichlet, uniform-boundary grid.

SD2: Dirichlet, local-nesting.

SD3: Dirichlet, piecewise continuous integration.

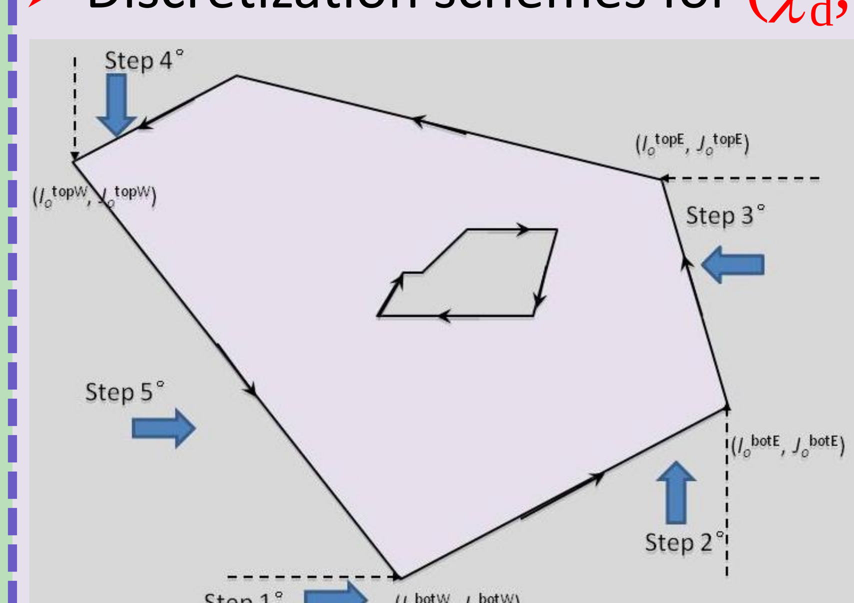
SN2: Neumann, local-nesting.

ψ_e^{true}	Scheme	SCC	RRD
(1)	SD1	1.000	0.485%
	SD2	1.000	0.015%
	SD3	1.000	0.11%
	SN2	0.981	20.7%
(2)	SD1	1.000	0.822%
	SD2	1.000	0.060%
	SD3	1.000	0.25%
	SN2	0.728	70.5%

Exp 2: $\psi_e^{true} = -8xy(x^2 - y^2) - 3(3x^2 - y^2)y - 8xy - 5y$ ... (1)

$\psi_e^{true} = \sum_m[a_m\sin(m\pi x/L)sh(m\pi y/L) + b_m sh(m\pi x/L)\sin(m\pi y/L)]$ ... (2)

➢ Discretization schemes for $(\chi_d,\ \psi_d)$: to format the B.C for integration.



Step1: Starts from the SW corner point and search eastward point-by-point along each row and northward row-by-row for 1st southernmost data point, (I_o^{botW}, J_o^{botW}).

Step2: Starts from (I_o^{botW}, J_o^{botW}) and search northward for (I_o^{botE}, J_o^{botE}).

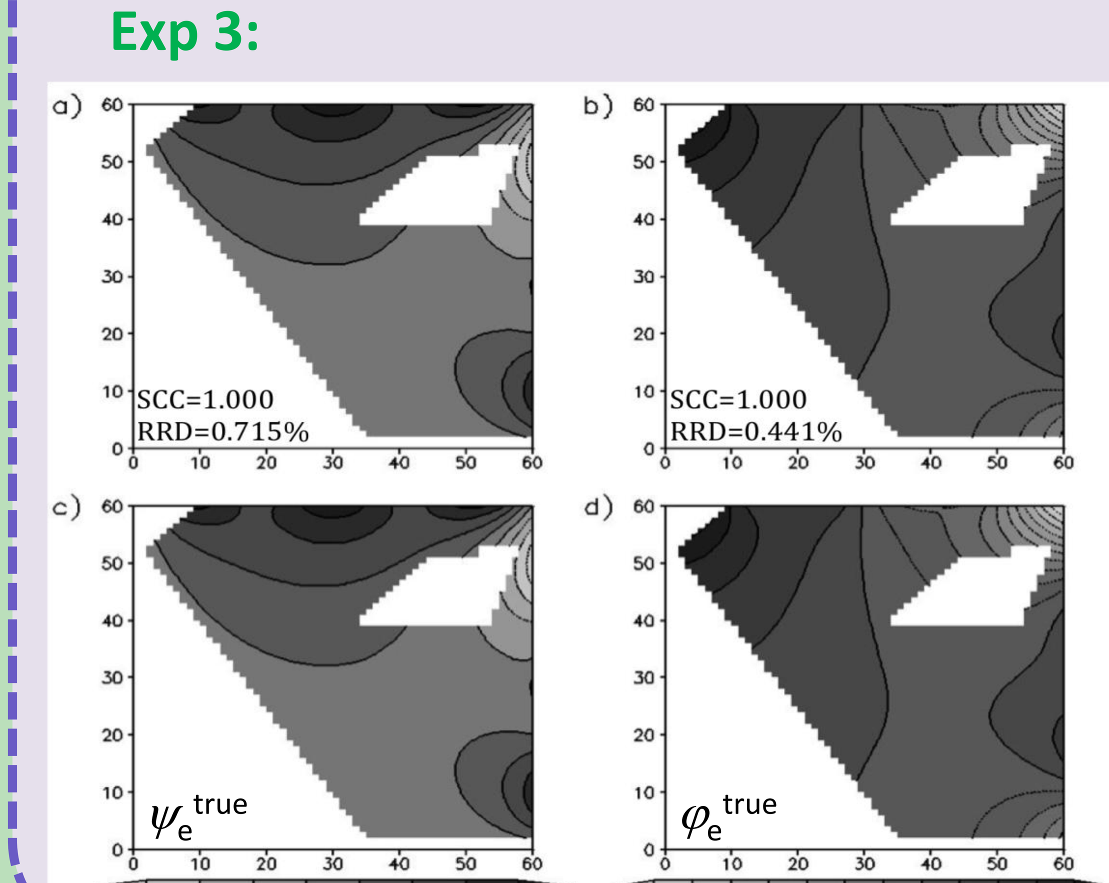
Step3: for (I_o^{topE}, J_o^{topE}).

Step4: for (I_o^{topW}, J_o^{topW}).

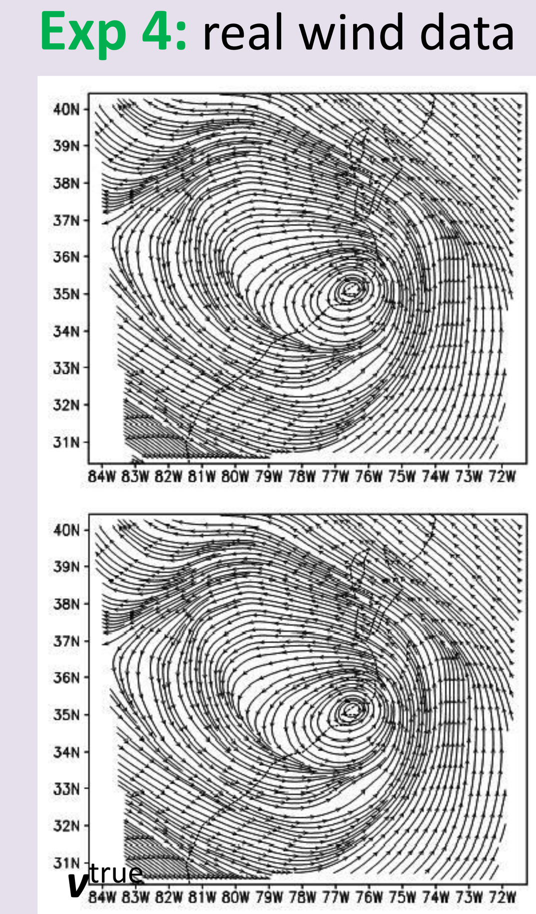
Step5: search back to (I_o^{botW}, J_o^{botW}).

Dash arrow: the starting point of each searching; shaded arrow: search direction.

Exp 3:



Exp 4: real wind data



4. Conclusions & Discussions

Accuracy. √

Irregular domains. √

Xu Q., **J. Cao**, and S. Gao, 2011: Computing streamfunction and velocity potential in a limited domain. Part I: Theory and integral formulae. *Adv. Atmos. Sci.* (SCI), **28**(6), 1433-1444.

Cao, J., and Xu Q., 2011: Computing streamfunction and velocity potential in a limited domain. Part II: Numerical methods and test experiments. *Adv. Atmos. Sci.* (SCI), **28**(6), 1445-1458.

Please email *iamcaojie@gmail.com* for codes, or questions.