

Solitary wave trains and internal undular bores

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in collaboration with:

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EGU 2019

A: Variable-coefficient Korteweg-de Vries equation

The depth h , and the background current u_0 , density ρ_0 vary slowly in the horizontal direction with x . Then the vertical particle displacement is $\zeta \sim \eta(x, t)\phi(z; x)$ where

$$\eta_t + c\eta_x + \frac{cQ_x}{2Q}\eta + \mu\eta\eta_x + \lambda\eta_{xxx} + \sigma\eta = 0,$$

$$\begin{aligned} \{\rho_0(c - u_0)^2\phi_z\}_z - g\rho_{0z}\phi &= 0, \quad \text{for } -h < z < \eta_0, \\ \phi &= 0 \quad \text{at } z = -h, \quad (c - u_0)^2\phi_z = g\phi \quad \text{at } z = \eta_0. \end{aligned}$$

The coefficients μ, λ, Q, σ are given by

$$I\mu = 3 \int_{-h}^{\eta_0} \rho_0 (c - u_0)^2 \phi_z^3 dz, \quad I\lambda = \int_{-h}^{\eta_0} \rho_0 (c - u_0)^2 \phi^2 dz,$$

$$I = 2 \int_{-h}^{\eta_0} \rho_0 (c - u_0) \phi_z^2 dz, \quad Q = c^2 I.$$

$$I\sigma = - \int_{-h}^{\eta_0} \phi\phi_z F_{0z} dz, \quad F_0 = \rho_0(u_0 u_{0x} + w_0 u_{0z}) + p_{0x}.$$

B. Canonical vKdV equation

The “spatial” evolution form is

$$A = Q^{1/2}\eta, \quad A_T + \frac{\nu}{Q^{1/2}}AA_X + \delta A_{XXX} + \sigma A = 0.$$

$$X = \int_{x_0}^x \frac{dx}{c} - t, \quad T = \int_{x_0}^x \frac{dx}{c}, \quad \nu = \frac{\mu}{c}, \quad \delta = \frac{\lambda}{c^3}.$$

A further transformation yields the canonical form

$$U_\tau + \alpha U U_X + U_{XXX} = 0,$$

$$\text{where } A = RU, \quad R = \exp\left(-\int_0^\tau \beta d\tau'\right),$$

$$\tau = \int_0^T \delta dT, \quad \alpha = \frac{R\nu}{\delta Q^{1/2}}, \quad \beta = \frac{\sigma}{\delta}.$$

R, α, β vary with τ . There are two conservation laws, the vKdV equation for mass and wave action flux,

$$\{U^2\}_\tau + \{2\alpha U^3/3 + 2UU_{XX} - U_X^2\}_X = 0.$$

C: Solitary wave train

A modulated solitary wave train is

$$U_{sol} = a \operatorname{sech}^2(\gamma\theta) + d, \quad \theta_X = k, \quad \theta_\tau = -kV,$$
$$V - \alpha d = \frac{\alpha a}{3} = 4\gamma^2 k^2.$$

with **three parameters**, the amplitude a , the pedestal d and the wavenumber k . The modulation equations are

$$d_\tau + \alpha d d_X = 0,$$

expressing conservation of mass. Hence d can be regarded as a known quantity. Conservation of wave action leads to

$$\mathcal{A}_\tau + \left(\alpha d + \frac{\alpha a}{3}\right)\mathcal{A}_X + \mathcal{A}\alpha d_X = 0, \quad \mathcal{A} = \left\{\frac{a^3}{\alpha}\right\}^{1/2}.$$

This is a nonlinear hyperbolic equation for \mathcal{A} alone.

Conservation of waves is the third equation,

$$k_\tau + (kV)_X = 0, \quad \text{where} \quad V = \alpha d + \frac{\alpha a}{3},$$

D: Solitary wave train similarity solutions

$\alpha > 0$ **Development of undular bore:**

$$d = 0, \quad a = \frac{3\alpha^{1/3}X}{\chi} > 0, \quad \chi = \int_0^\tau \alpha^{4/3}(\tau') d\tau'.$$

This holds over the domain $X_m(\tau) < X < X_M(\tau)$ say. At the head the amplitude is $a_M = 3\alpha^{1/3}X_M/\chi$, and at the rear the amplitude is $a_m = 3\alpha^{1/3}X_m/\chi$.

$\alpha < 0$ **Rarefaction wave with following undular bore:**

$$d = \frac{X - X_0}{\eta}, \quad \eta = \int_{\tau_c}^\tau \alpha(\tau') d\tau', \quad X < X_0,$$
$$a = -\frac{3|\alpha|^{1/3}X}{\eta\xi} < 0, \quad \xi = \int_{-\infty}^\eta \frac{|\alpha(\eta')|^{1/3} d\eta'}{|\eta'|^{5/3}},$$

Since $\alpha < 0$, $\eta < 0$, ensuring that the rarefaction wave $d > 0, a < 0$ in $X < X_0$.

1. Internal solitary waves: Ocean

Internal tide and solitary waves, Pacific North West

An Atlas of Oceanic Internal Solitary Waves (May 2002)
by Global Ocean Associates
Prepared for the Office of Naval Research - Code 322PO

Oregon Coast - Columbia River

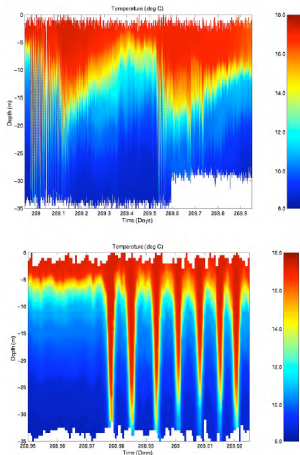


Figure 2. (Upper) A color contour time series of temperature profiles from the surface to 35m depth measured by the LMP over a one-day period. The 10°C span color contour scale is shown the right of the time series panel. The low frequency, semidiurnal internal tide displacement can clearly be seen along the yellow isotherm. (Lower) A profile time series of the first 1.7 hours of the time series shown in Figure 1a. White areas indicate times with no data. [From Stanton and Ostrovsky, 1998]

2. Internal solitary waves: Ocean

Satellite view, Pacific North West

An Atlas of Oceanic Internal Solitary Waves (May 2002)
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Oregon Coast - Columbia River

Oregon Coast - Colombia River

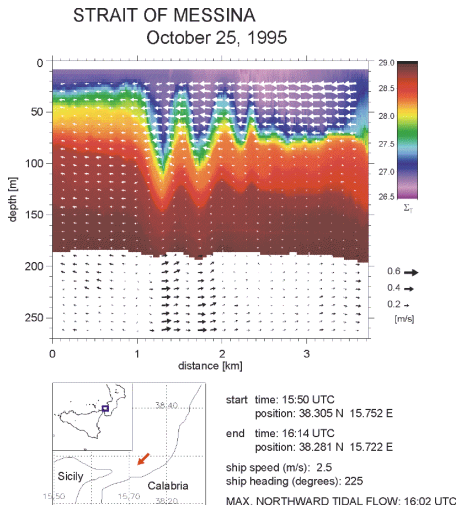
with co-authors T. P. Stanton; Department of Oceanography, Naval Postgraduate School and
L. A. Ostrovsky CIRES, University of Colorado, NOAA ETL, CO 80303



Figure 1. Radarsat-1 image of the Columbia River and Oregon coast showing internal waves. The river plume induces seaward propagating internal waves. The shoreward propagating waves were generated at the coastal shelf break.

3. Internal solitary waves: Ocean

Strait of Messina



4. Internal solitary waves: Ocean

Strait of Messina, satellite view

An Atlas of Oceanic Internal Solitary Waves (February 2004)
by Global Ocean Associates
Prepared for Office of Naval Research – Code 322 PO

Strait of Messina

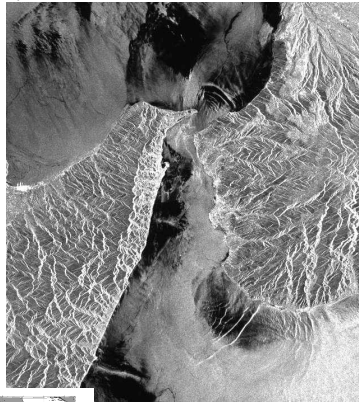


Figure 5 ERS-1 (C-band VV) SAR image of the Strait of Messina acquired on 11 July 1993 at 0941 UTC (orbit 10387, frame 2835). The image shows internal wave signatures radiating out of the strait in both the northern and southern directions. Northwards propagating internal waves are less frequently observed than southward propagating ones. Imaged area is 65 km x 65 km. CESA 1993. [From The Tropical and Subtropical Ocean Viewed by ERS SAR <http://www.ifm.uni-hamburg.de/ers-sar/>]

5. Internal solitary waves: Ocean

Strait of Gibraltar satellite view

An Atlas of Oceanic Internal Solitary Waves (February 2004)
by Global Ocean Associates
Prepared for Office of Naval Research – Code 322 PO

Strait of Gibraltar

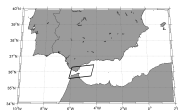


Figure 12. Astronaut photograph (STS41G-34-81) of Gibraltar region acquired on 11 October 1984 at 1222 UTC. The image shows three packets of solitons to the east of Gibraltar. Older packets are visible along Spanish and Moroccan coasts. [Image courtesy of Earth Sciences and Image Analysis Laboratory, NASA Johnson Space Center (<http://eol.jsc.nasa.gov/>)]



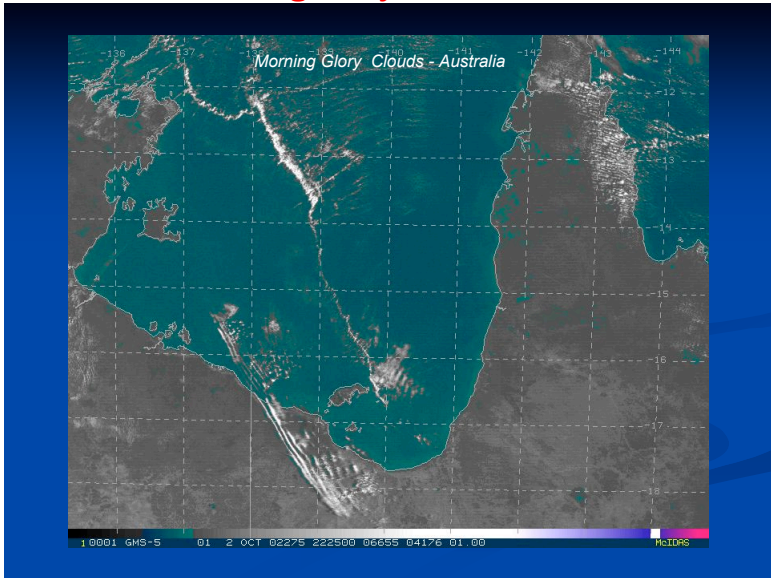
6. Internal solitary waves: Atmosphere

Morning Glory Waves



7. Internal solitary waves: Atmosphere

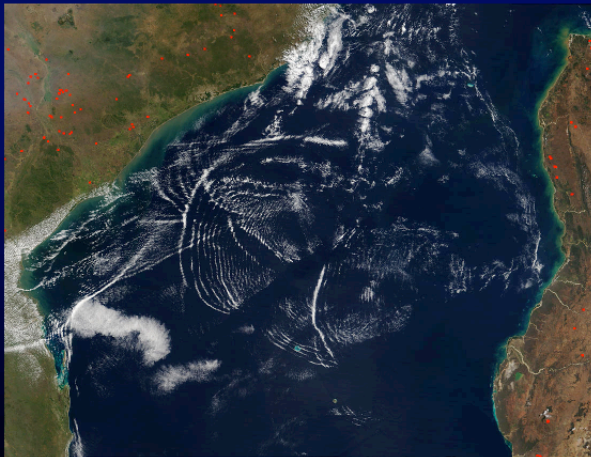
Satellite view of Morning Glory Waves



8. Internal solitary waves: Atmosphere

Satellite view of Mozambique waves

Atmospheric internal waves



MODIS TERRA image Mozambique channel (16 AUG 2002)

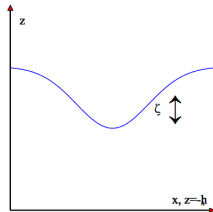
9. Internal solitary waves: Atmosphere

Sable Island, Canada



10. Weakly nonlinear model

Typically internal solitary waves are **long nonlinear** waves. Here we sketch the theory for the oceanic case.



The background state is the density $\rho_0(z)$ and horizontal current $u_0(z)$, while the **buoyancy frequency** is given by

$$\rho_0 N^2 = -g \rho_{0z} . \quad (1)$$

Then seek an asymptotic solution for the vertical particle displacement $\zeta = \zeta(x, z, t)$, based on the assumption that the waves are **long and weakly nonlinear**.

11. Korteweg-de Vries (KdV) equation

In a uniform medium this is given asymptotically by $\zeta \sim \eta(x, t)\phi(z)$, where

$$\eta_t + c\eta_x + \mu\eta\eta_x + \lambda\eta_{xxx} = 0. \quad (2)$$

$$\{\rho_0(c - u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0, \quad \text{for } -h < z < \eta_0, \quad (3)$$

$$\phi = 0 \quad \text{at } z = -h, \quad (c - u_0)^2\phi_z = g\phi \quad \text{at } z = \eta_0. \quad (4)$$

The coefficients μ, λ are given by

$$I\mu = 3 \int_{-h}^{\eta_0} \rho_0 (c - u_0)^2 \phi_z^3 dz, \quad I\lambda = \int_{-h}^{\eta_0} \rho_0 (c - u_0)^2 \phi^2 dz,$$

$$I = 2 \int_{-h}^{\eta_0} \rho_0 (c - u_0) \phi_z^2 dz. \quad (5)$$

The KdV equation (2) is integrable. We are concerned with the **undular bore** solution, the outcome of a step-like initial condition using the Whitham modulation equations.

12. Variable-coefficient KdV (vKdV) equation

Now suppose the depth h , and background current u_0 , density ρ_0 vary slowly in the horizontal direction with x . The KdV equation (2) is replaced by a similar equation, but with two extra terms,

$$\eta_t + c\eta_x + \frac{cQ_x}{2Q}\eta + \mu\eta\eta_x + \lambda\eta_{xxx} + \sigma\eta = 0, \quad Q = c^2I, \quad (6)$$

$$I\sigma = - \int_{-h}^{\eta_0} \phi\phi_z F_{0z} dz, \quad F_0 = \rho_0(u_0 u_{0x} + w_0 u_{0z}) + p_{0x}. \quad (7)$$

The modal equation now depends on x **parametrically**, that is $\phi = \phi(z; x)$, $c = c(x)$, and hence the coefficients μ , λ , Q also depend (slowly) on x . In (7) p_0 is the basic pressure, such that $p_{0z} = -g\rho_0$, and w_0 is the basic vertical velocity such that $u_{0x} + w_{0z} = 0$. The term $\sigma\eta$ arises when the basic state is not a solution of the inviscid Euler equations.

13. Canonical vKdV equation

The “spatial” evolution form, asymptotically equivalent to (6),

$$A = Q^{1/2}\eta, \quad A_T + \frac{\nu}{Q^{1/2}}AA_X + \delta A_{XXX} + \sigma A = 0. \quad (8)$$

$$X = \int_{x_0}^x \frac{dx}{c} - t, \quad T = \int_{x_0}^x \frac{dx}{c}, \quad \nu = \frac{\mu}{c}, \quad \delta = \frac{\lambda}{c^3}. \quad (9)$$

A further transformation yields the canonical form

$$U_\tau + \alpha UU_X + U_{XXX} = 0, \quad (10)$$

$$\text{where } A = RU, \quad R = \exp\left(-\int_0^\tau \beta d\tau'\right),$$

$$\tau = \int_0^T \delta dT, \quad \alpha = \frac{R\nu}{\delta Q^{1/2}}, \quad \beta = \frac{\sigma}{\delta}. \quad (11)$$

R, α, β vary with τ . There are two conservation laws, (10) for mass and wave action flux,

$$\{U^2\}_\tau + \{2\alpha U^3/3 + 2UU_{XX} - U_X^2\}_X = 0. \quad (12)$$

14. Solitary wave train

A modulated solitary wave is

$$U_{sol} = a \operatorname{sech}^2(\gamma\theta) + d, \quad \theta_x = k, \quad \theta_\tau = -kV, \quad (13)$$

$$V - \alpha d = \frac{\alpha a}{3} = 4\gamma^2 k^2. \quad (14)$$

with **three parameters**, the amplitude a , the pedestal d and the wavenumber k . This is an exact solution if these parameters are constant. Here, especially when $\alpha(\tau)$ is **slowly varying**, a, d, k are **slowly varying**. To find expressions for that variation, we use the Whitham modulation theory, developed for periodic waves, adapted for the solitary wave train, where (13) is regarded as sitting on a periodic lattice. Or one can use a direct multi-scale asymptotic expansion. Here we use the Whitham approach, using (10) and (12) for mass and wave action flux, together with the conservation of waves,

$$k_\tau + (kV)_x = 0. \quad (15)$$

15. Solitary wave train

The outcome is

$$d_\tau + \alpha d d_x = 0, \quad (16)$$

expressing conservation of mass. This is a Hopf equation for d alone, and so d can be regarded as a known quantity.

Conservation of wave action leads to

$$\mathcal{A}_\tau + \left(\alpha d + \frac{\alpha a}{3}\right) \mathcal{A}_x + \mathcal{A} \alpha d_x = 0, \quad \mathcal{A} = \left\{\frac{a^3}{\alpha}\right\}^{1/2}. \quad (17)$$

With d already determined, this is a nonlinear hyperbolic equation for \mathcal{A} alone, thus determining the amplitude a .

Conservation of waves is the third equation,

$$k_\tau + (kV)_x = 0, \quad \text{where} \quad V = \alpha d + \frac{\alpha a}{3}, \quad (18)$$

can be regarded as known. This is a linear hyperbolic equation.

16: Undular bore

When $\alpha > 0$ **is a constant**, an undular bore can be found as an asymptotic solution of the KdV equation. This is an expanding wave train connecting a zero level at the front to a mean level $U_0 > 0$. **The leading wave is a solitary wave of amplitude $2U_0$.**

However, in a variable medium, $\alpha = \alpha(\tau)$, if an undular bore retains its structure as **a single-phase wave train**, then the **jump U_0 is preserved**, and so then the leading solitary wave would have a constant amplitude $2U_0$. But this is **inconsistent** with the result that the leading solitary wave amplitude should behave as $|\alpha|^{1/3}$. The resolution of this inconsistency is the formation of **a solitary wave train ahead of the undular bore**. This solitary wave train is described by (16, 17, 18), and there is a region where the rear of the solitary wave train interacts with the undular bore, forming a two-phase wave interaction.

17: Simulations

$$U_\tau + \alpha U U_X + U_{XXX} = 0,$$

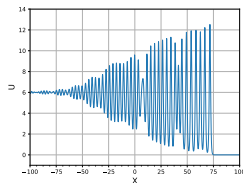
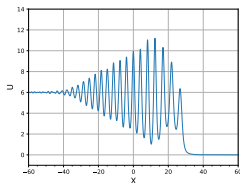
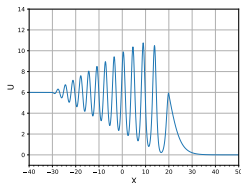
$$\alpha = 1 + (\alpha_a - 1) \tanh(K\tau), \quad \alpha(\tau = 0) = 1 \quad \alpha(\tau \rightarrow \infty) = \alpha_a.$$

Either $\alpha_a > 1$ for propagation up a slope, or $\alpha_a < 0$ for propagation up a slope and through a **critical point of polarity change where $\alpha = 0$** . The initial condition $U(X, 0) = U_{ic}(X)$ is a modulated cnoidal wave representation of an undular bore in the constant coefficient KdV equation evolving from a step of height $U_0 > 0$, $\tau = \tau_1$,

$$U_{ic}(X) = U_0 \text{ENV}(X) \{2m \text{cn}^2(\kappa(X - V\tau_1); m) + 1 - m\},$$
$$-U_0\tau_1 < X < \frac{2U_0\tau_1}{3}, \quad V = \frac{U_0}{3} \{1 + m\}, \quad U_0 = 6\kappa^2 k^2,$$
$$X = \frac{U_0\tau_1}{3} \left\{ 1 + m - \frac{2m(1-m)K(m)}{E(m) - (1-m)K(m)} \right\}.$$

ENV(X) is a box of height 1, containing the initial bore. 

18: Simulation: Undular bore on a slope



A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from 1 to 1.5 for the undular bore initial condition with $U_0 = 6$, $\tau_1 = 5$ and $K = 0.3$, $\tau_a = 10$ ($K\tau_a = 3$); the left panel is at $\alpha = 1$, the middle panel is at $\alpha = 1.25$ and the right panel is at $\alpha = 1.5$.

19: Solitary wave train interpretation

For the leading waves, seek a similarity solution of (16, 17, 18),

$$d_\tau + \alpha d d_\chi = 0,$$

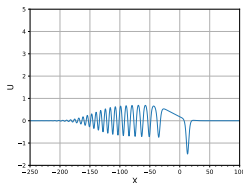
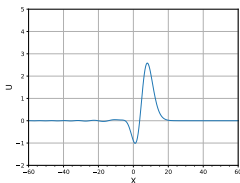
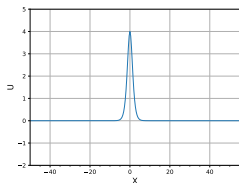
$$\mathcal{A}_\tau + \left(\alpha d + \frac{\alpha a}{3}\right) \mathcal{A}_\chi + \mathcal{A} \alpha d_\chi = 0, \quad \mathcal{A} = \left\{ \frac{a^3}{\alpha} \right\}^{1/2}.$$

$$k_\tau + (kV)_\chi = 0.$$

$$d = 0, \quad a = \frac{3\alpha^{1/3} X}{\chi} > 0, \quad \chi = \int^\tau \alpha^{4/3}(\tau') d\tau'. \quad (19)$$

This holds over the domain $X_m(\tau) < X < X_M(\tau)$ say. At the head the amplitude is $a_M = 3\alpha^{1/3} X_M / \chi$, and at the rear the amplitude is $a_m = 3\alpha^{1/3} X_m / \chi$. This must be matched to the following undular bore to determine $X_{M,m}(\tau)$. Note that if it is assumed that the leading wave in the following undular bore has amplitude $2U_0$ then we may expect that $a_m \approx 2U_0$ and $a_M \approx 2U_0 \alpha^{1/3}$, since initially $\alpha = 1$.

20: Simulation of a polarity change for a single solitary wave



A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from 1 to -1 for a **solitary wave initial condition** with $U_0 = 4$ and $K = 0.03$, $\tau_a = 100$ ($K\tau_a = 3$); the left panel is at $\alpha = 1$, the middle panel is at $\alpha = 0$ and the right panel is at $\alpha = -1$. Note the solitary waves moving forward along the rarefaction.

21: Polarity change interpretation

There is a another similarity solution of (16, 17, 18) for this case of polarity change, **after the critical point**,

$$d_\tau + \alpha d d_X = 0,$$

$$\mathcal{A}_\tau + (\alpha d + \frac{\alpha a}{3}) \mathcal{A}_X + \mathcal{A} \alpha d_X = 0, \quad \mathcal{A} = \left\{ \frac{a^3}{\alpha} \right\}^{1/2}.$$

$$k_\tau + (kV)_X = 0.$$

$$d = \frac{X - X_0}{\eta}, \quad \eta = \int_{\tau_c}^{\tau} \alpha(\tau') d\tau', \quad X < X_0, \quad (20)$$

$$a = -\frac{3|\alpha|^{1/3} X}{\eta \xi} < 0, \quad \xi = \int_{-\infty}^{\eta} \frac{|\alpha(\eta')|^{1/3} d\eta'}{|\eta'|^{5/3}}, \quad (21)$$

Here $\alpha < 0$ and so $\eta < 0$, ensuring that the rarefaction wave $d > 0$ in $X < X_0$. The determination of X_0 requires a detailed matching with the solution at the critical point, omitted here.

22: Polarity change interpretation

The rarefaction wave (20) can only extend to a point $X - X_0 = -L_r(\eta)$ where $L_r(\eta)$ is likewise undetermined. But the **mass of the rarefaction wave** is then $-L_r^2(\eta)/2\eta$ and this can be approximately equated to **the initial solitary wave mass** $2U_0/\kappa = 2(12U_0)^{1/2}$, thus giving an approximate expression for $L_r(\eta)$. The expression (21) for the solitary wave amplitude a holds on the domain $-L_r(\eta) < X - X_0 < -L_s(\eta)$ where the upper bound $L_s(\eta)$ determines the amplitude of the leading solitary wave, that is $a_s = -3|\alpha|^{1/3}L_s/|\eta|^{5/3}\xi$. The values of a_s, L_s are undetermined and requires matching with the solution at the critical point. However, an approximate estimate can be based on the assumption that since the emerging solitary wave train is the leading edge of an undular bore resolving the jump at the rear of the rarefaction wave, and then $a_s = 2L_r/\eta$, where in turn L_r is estimated from conservation of mass, as above.

23: Simulation: Polarity change for an undular bore

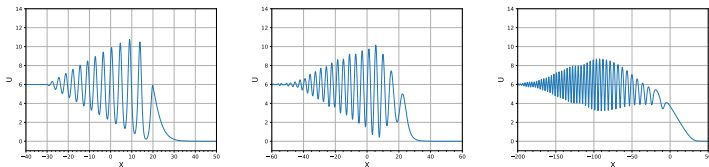


Figure: A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from 1 to -1 for the undular bore initial condition with $U_0 = 6$, $\tau_1 = 5$ and $K = 0.3$, $\tau_a = 10$ ($K\tau_a = 3$); the left panel is at $\alpha = 1$, the middle panel is at $\alpha = 0$ and the right panel is at $\alpha = -1$.

24: Summary

$$d_\tau + \alpha d d_X = 0,$$

$$\mathcal{A}_\tau + \left(\alpha d + \frac{\alpha a}{3}\right) \mathcal{A}_X + \mathcal{A} \alpha d_X = 0, \quad \mathcal{A} = \left\{ \frac{a^3}{\alpha} \right\}^{1/2},$$

$$k_\tau + (kV)_X = 0, \quad \text{where} \quad V = \alpha d + \frac{\alpha a}{3},$$

The solitary wave train equations are a very powerful tool for analysing solitary waves and undular bores in variable media, especially because they are each hyperbolic equations which can be solved in sequence for d , \mathcal{A} , k .

Although developed here for the KdV equation, I expect they can be found for other nonlinear wave equations.

Thank you: References

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