Numerical solution of the mass continuity equation for snowpack modeling on moving meshes:

Coupling between mechanical settling and water (vapor) transport

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Motivation

Well known: The processes mechanical settling and phase change are coupled in snowpacks

Alpine snowpacks


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Snow avalanche risk assessment

Polar snowpacks


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Climate models

Current snowpack models lack a sound mathematical coupling of both processes

With such a model we could:

- assess competing effects from mechanical settling and phase change in the snowpack
- improve representation of snow properties

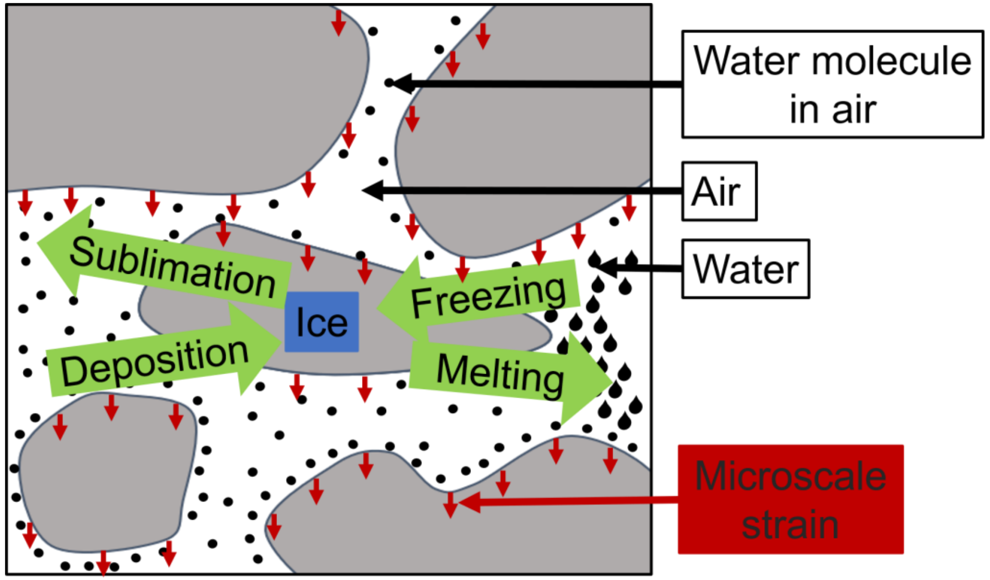
Ice Mass Balance – Starting Point for a Flexible Snowpack Model Including Settling

The ice phase evolves due to coupled mechanical and metamorphic phase change processes

$$\partial_t \phi + \partial_z(\phi \cdot v) = \frac{1}{\rho_i} c$$

Microscale processes are captured in macroscale properties

Microscale



Macroscale

- **Ice volume fraction ϕ**: ice volume per total volume
- **Phase change rate c**: loss or gain of ice mass in a specific volume per time
- **Settling velocity v**: settlement due to mechanical strain per time

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Ice Mass Balance – Starting Point for a Flexible Snowpack Model Including Settling

The snowpack evolves due to
coupled mechanical and metamorphic phase change processes

$$\boldsymbol{\partial_t \phi + \partial_z(\phi \cdot v) = \frac{1}{\rho_i} c}$$

ice volume fraction $\boldsymbol{\phi}$ **settling velocity $\boldsymbol{v}$** **phase change rate $\boldsymbol{c}$**

Challenge:

- **Flexible solution technique –** Solve the ice mass balance for ice volume fraction in a way that can be applied to generic settling velocities and phase change processes
- **Settling velocity –** Parametrize the settling velocity in a physically consistent way
- **Metamorphic phase change –** Couple settling to complex phase change operators that result from established snow and firn models

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Flexible Solution Technique – General Idea

The snowpack evolves due to coupled mechanical and metamorphic phase change processes

$$\partial_t \phi + \partial_z(\phi \cdot v) = \frac{1}{\rho_i} c$$

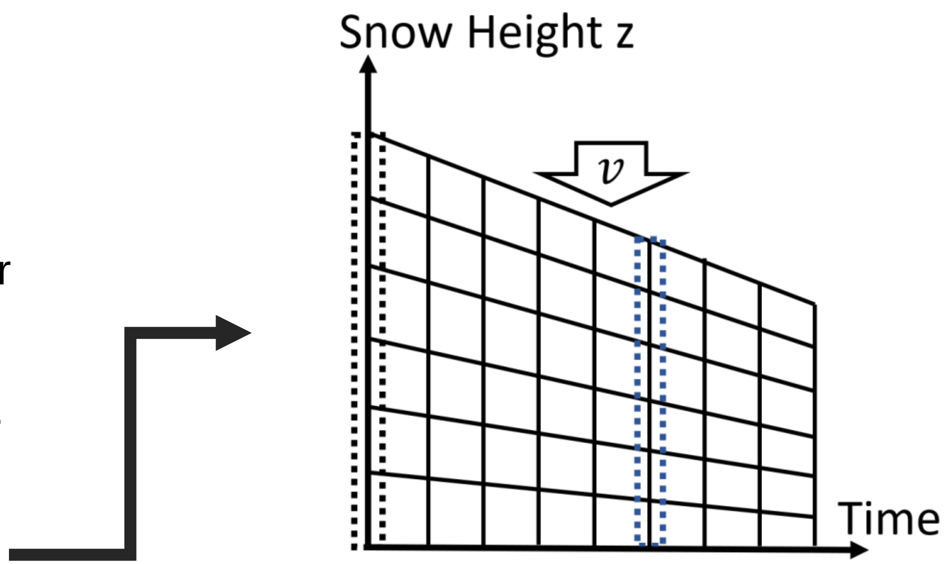
Two-step approach:

Step 1: Determine **phase change rate** from a conventional process model in a **Eulerian** reference frame,

e.g. solution for dry snow in model from Hansen and Foslien (2015) (or other process models such as Calonne et al. (2014))

Step 2: Use **phase change rate** to solve ice mass balance for **ice volume fraction** based on a **settling velocity** in a **Lagrangian** reference frame,

e.g. a mesh strain based on the method of characteristics



Note: Due to Step 2 the mesh will be distorted, hence Step 1 has to be solved on a non-uniform grid!

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Flexible Solution Technique - Mixed Eulerian Lagrangian Solution Method

Method of Characteristics (MOC) to solve non-linear Advection Equation with Source Term

Ice Mass Balance $\partial_t \phi + \partial_z(\phi \cdot v) = \frac{1}{\rho_i} c$ → Apply MOC → Update of **ice volume fraction** with coupled coordinate update

$$\partial_t z = v$$
$$\partial_t \phi = \frac{1}{\rho_i} c + \partial_z v \cdot \phi$$

Analytical

For $c = const$ and $v = const$ exists an analytical solution

$$\phi = \frac{1}{\rho_i} c \cdot t + \phi^0 \qquad \text{because } \partial_z v(\phi, z, \eta) = 0$$
$$z = v \cdot t + z^0$$

Simple constant settling velocity closure only

Numerical

Let $\phi_k^n := (\, t_n, z_k)\; t_n, n\, \epsilon \{\, 0, \dots, N\}$ be a discretization of time axis t and $z_k, k\, \epsilon \{0, \dots, K\}$ be a spatial discretization. Then $\phi_k^n := \phi(t_n, z_k)$

$$\phi_k^{n+1} = \phi_k^n + \Delta t \cdot \left(\frac{1}{\rho_i} c_k^n + \partial_z v \cdot \phi_k^n \right)$$
$$z_k^{n+1} = z_k^n + \Delta t \cdot v_k^n$$

Flexibility: **arbitrary** settling velocity closures

From the perspective of common snowpack models:
Consider MOC as extension of the „layer boundary motion scheme“ that combines settling with the source term

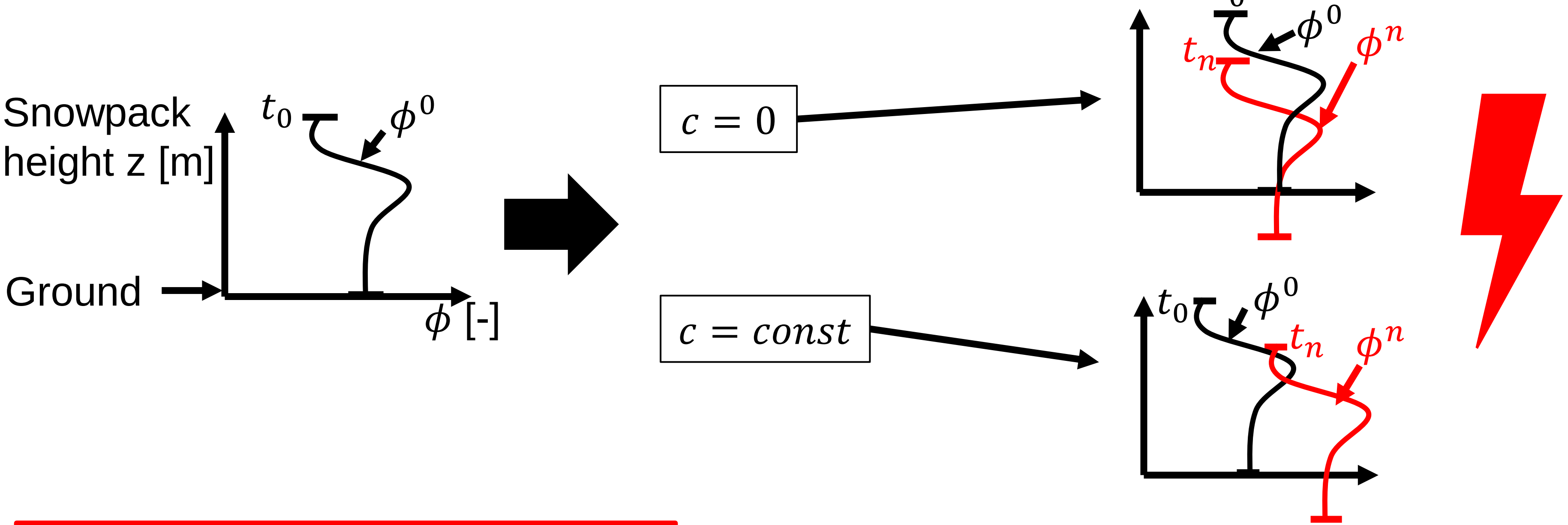
Legend

Settling Velocity – Simple Constant Velocity Closure Leads to Non-realistic Results

Constant settling velocity yields linear advection equation

$$\partial_t \phi + v \partial_z(\phi) = c$$

Initial ice volume fraction: $\phi(t_0, z) = \phi^0$



Snowpack penetrates ground:
$v = const$ does not comply with reality!

Physical constraints have to be incorporated!

Legend

Settling Velocity – Physical Constraints for Realistic Settling

Settling process in a snowpack model has to comply with the following physical constraints:

1) **Non-penetration of the ground**,
   hence the settling velocity has to vanish at height zero
2) **Incompressibility of ice**,
   or rather compressibility is only due to a change in volume fraction, such that the snowpack can densify only up to a maximum given value
3) **Self-consolidation**,
   velocity at location z is dependent on all strain below z, hence the settling velocity is given by the integral of the local strain rate

Reflect physical constraints in settling velocity equations!

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Settling Velocity – Connect Self-consolidation with Local Strain

Settling velocity is the integral of the local strain rate:

Local Strain [-]		$\dot{\varepsilon}(z)$: Local Strain rate [s^{-1}]		$v(z)$: Local Settling Velocity [ms^{-1}]
Local height change divided by total height	Time rate of change →	$\dot{\varepsilon}(z) = \partial_z v(z)$	Integrate over height z →	$v(z) = \int_0^z \dot{\varepsilon}(\tilde{z})\, d\tilde{z}$

Observation:
Settling velocity as integrated from the local strain rate,
is inherently non-penetrating,
hence complies with constraint 1)

Strategy:
Test Mixed Eulerian Lagrangian solution method for a number of strain rate closures

1) Test concept with several strain rates and 2) use them in equation for settling velocity

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Settling Velocity – Hierarchy of Test Cases

Strain rates $\dot{\varepsilon}$ of increasing complexity are integrated to settling veloctities $\boldsymbol{v}$

Depth dependent

$$\dot{\varepsilon}(z) = D_c(z)$$

$$v(z) = \int_0^z \dot{\varepsilon}(\tilde{z})\, d\tilde{z}$$

- Strain rate is depth dependent
- D_c : strain rate coefficient
- Velocity independent of ice volume fraction
- Used for numerical benchmark

Ice volume dependent

$$\dot{\varepsilon}(\phi(z)) = \boldsymbol{D_c} \cdot \left(\boldsymbol{1} - \frac{\boldsymbol{\phi(z)}}{\boldsymbol{\phi_{max}(z)}}\right)$$

$$v\big(\phi(z)\big) = \int_0^z \dot{\varepsilon}\big(\phi(\tilde{z}\,)\big) d\tilde{z}$$

- Strain rate depends on ice volume fraction $\boldsymbol{\phi}$
- D_c : strain rate coefficient
- Commonly used in firn models[1]

Stress dependent

$$\dot{\varepsilon}(\sigma(z), \eta(T)) = \frac{\boldsymbol{1}}{\boldsymbol{\eta(T)}} \boldsymbol{\sigma(\phi, z)}$$

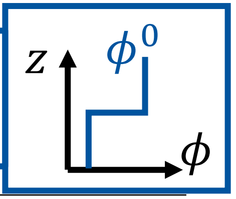
$$v(\sigma(\phi, z), \eta(T)) = \int_0^z \dot{\varepsilon}(\ldots) d\tilde{z}$$

- Strain rate depends on stress $\boldsymbol{\sigma}$ and temperature dependent viscosity $\boldsymbol{\eta}$
- Derived from constitutive equation for mechanics
- Commonly used in snow models[2]

Neglect phase change rate $c = 0$

Run simulations with strain rate/settling velocity pairs

Initial ice volume fract. $\boldsymbol{\phi^0}$: 2 layer snowpack



Legend

[1] Lundin et al. (2017)
[2] Vionnet et al. (2012), Lehning et al. (2002)

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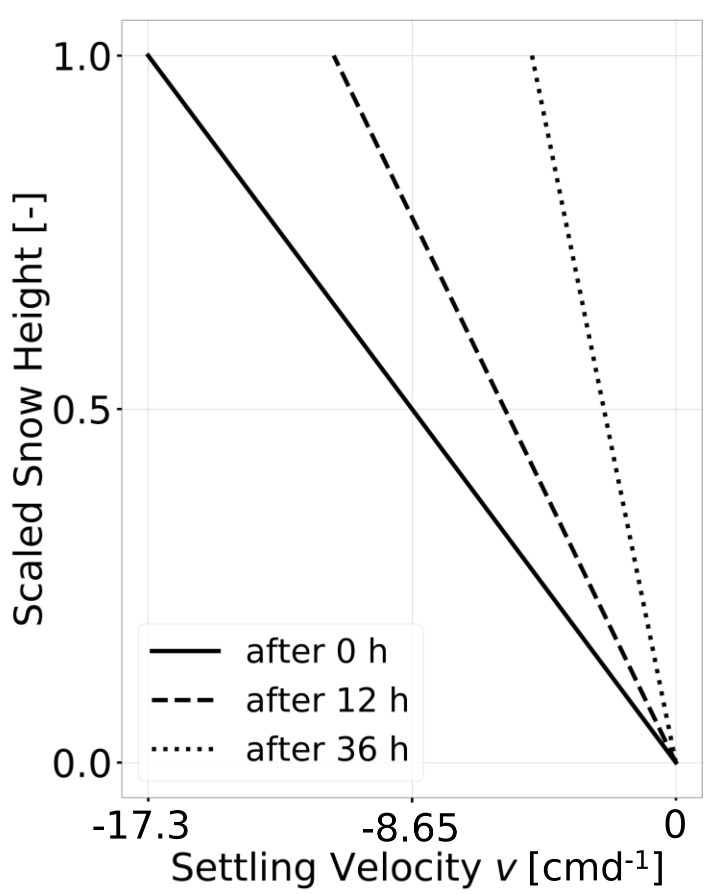
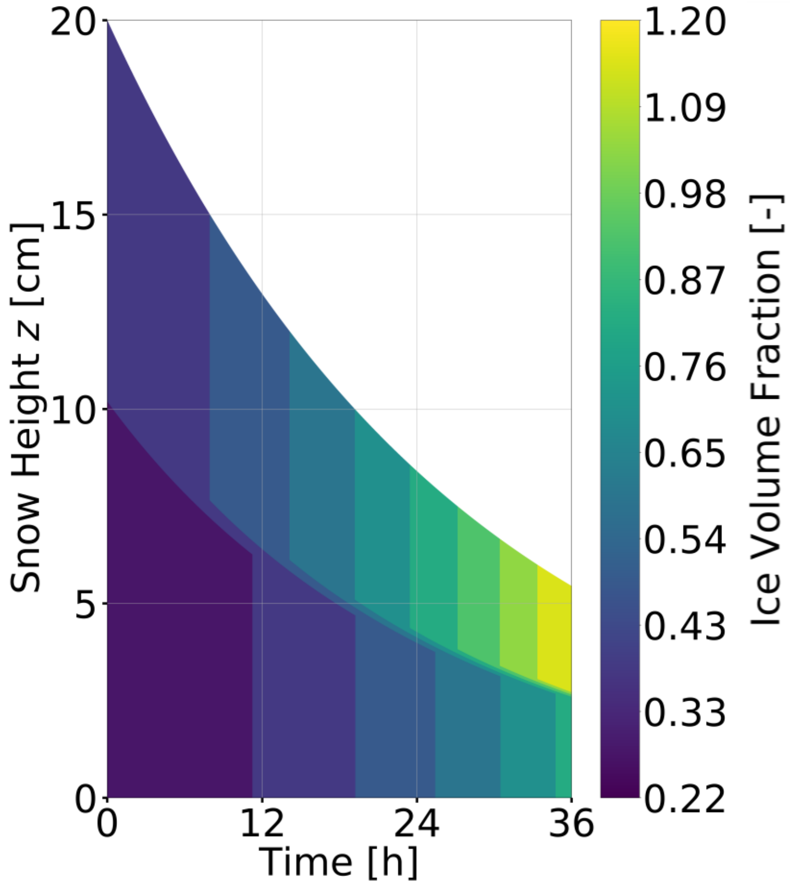
Simulation Results – Depth Dependent Settling

Constant strain rate coefficient $D_c = 10^{-5} s^{-1}$ [1]





We observe that:

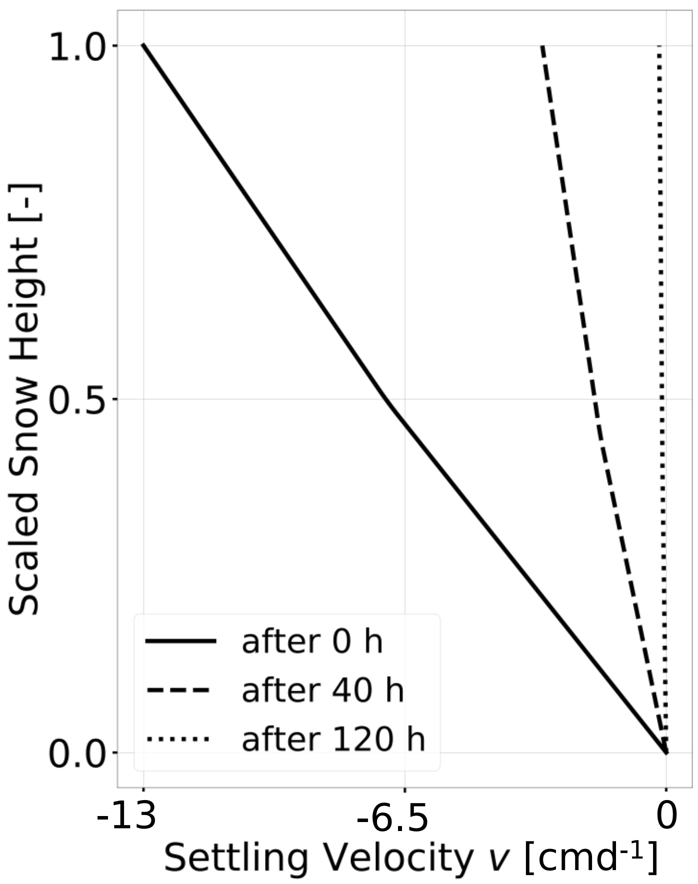
- **Snow height**: decreases continuously
- **Settling velocity:** linear and decreases with time
- **Layer thickness:** upper layer decreases faster
- **Ice volume fraction:** increases to non-physical value above 1

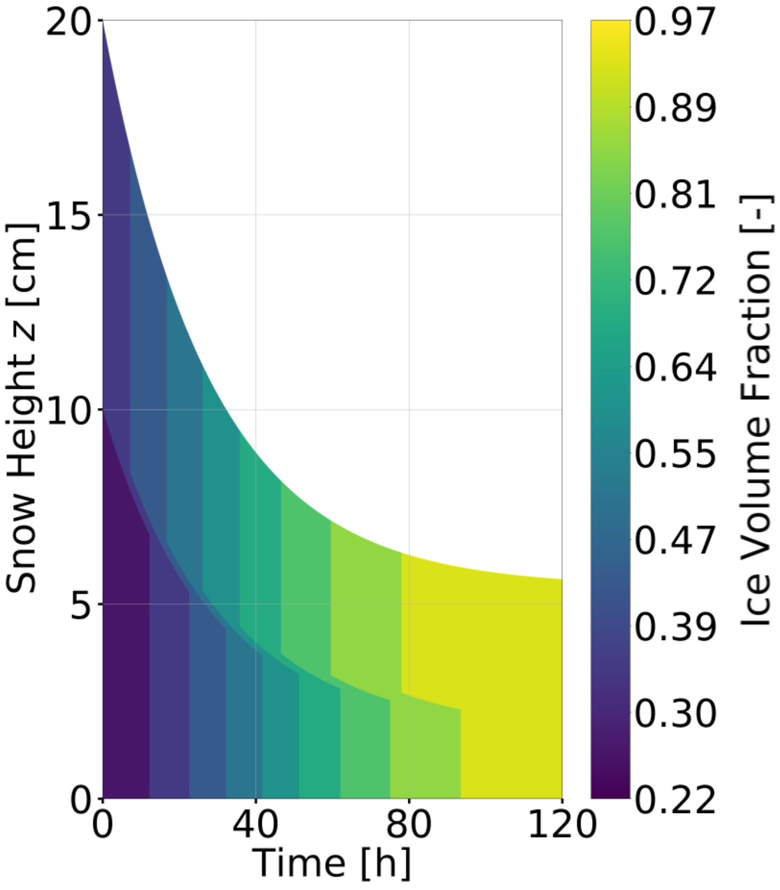
✓ Non-penetration
✗ Incompressibility
✓ Self-consolidation

[1] Intermediate value of fast (>$10^{-4} s^{-1}$) and slow (<$10^{-6} s^{-1}$) strain rates (Johnson (2011))

Simulation Results – Ice Volume Fraction Dependent Strain Rate

Maximum ice volume fraction $\phi_{max} = 1$





We observe that:

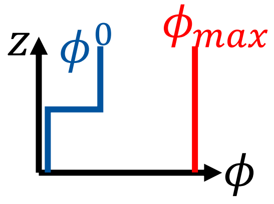
- **Snow height**:
  decreases with realistic asymptote
- **Settling velocity:**
  piecewise linear and decreasing
- **Layer thickness:**
  Lower layer decreases faster
- **Ice volume fraction:**
  increases to ϕ_{max}

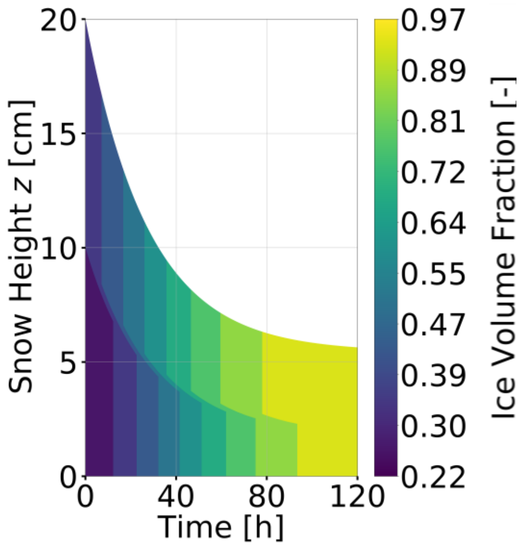
✓ Non-penetration
✓ Incompressibility
✓ Self-consolidation

Simulation Results – Ice Volume Fraction Dependent Strain Rate

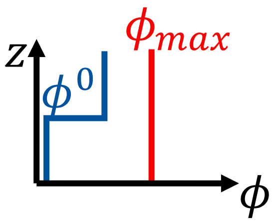
Our approach allows a flexible depth dependent definition of $\phi_{max}(z)$

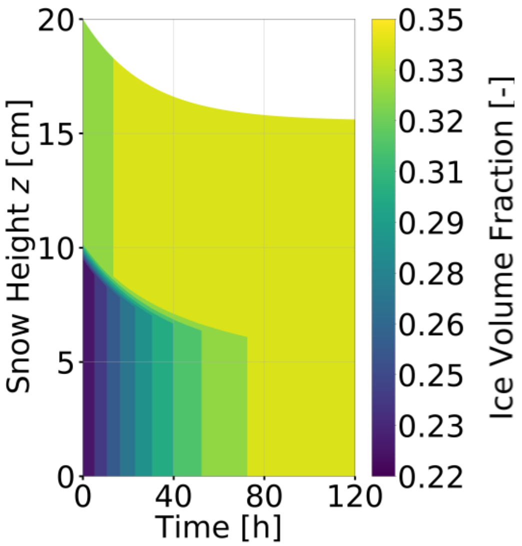
$\phi_{max} = 0.9 \gg \phi^0$



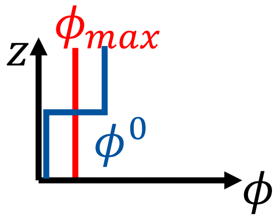


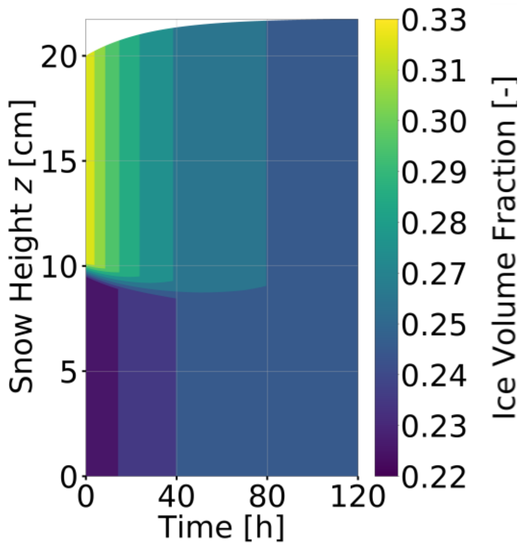
$\phi_{max} = 0.35 > \phi^0$



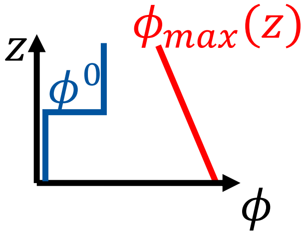


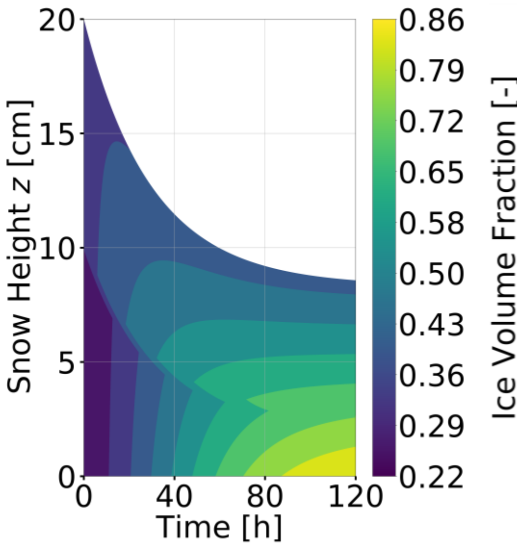
$\phi_{max} = 0.25 < \phi^0$





$\phi_{max}(z) = \frac{0.4-0.9}{H} \cdot z + 0.9 > \phi^0$





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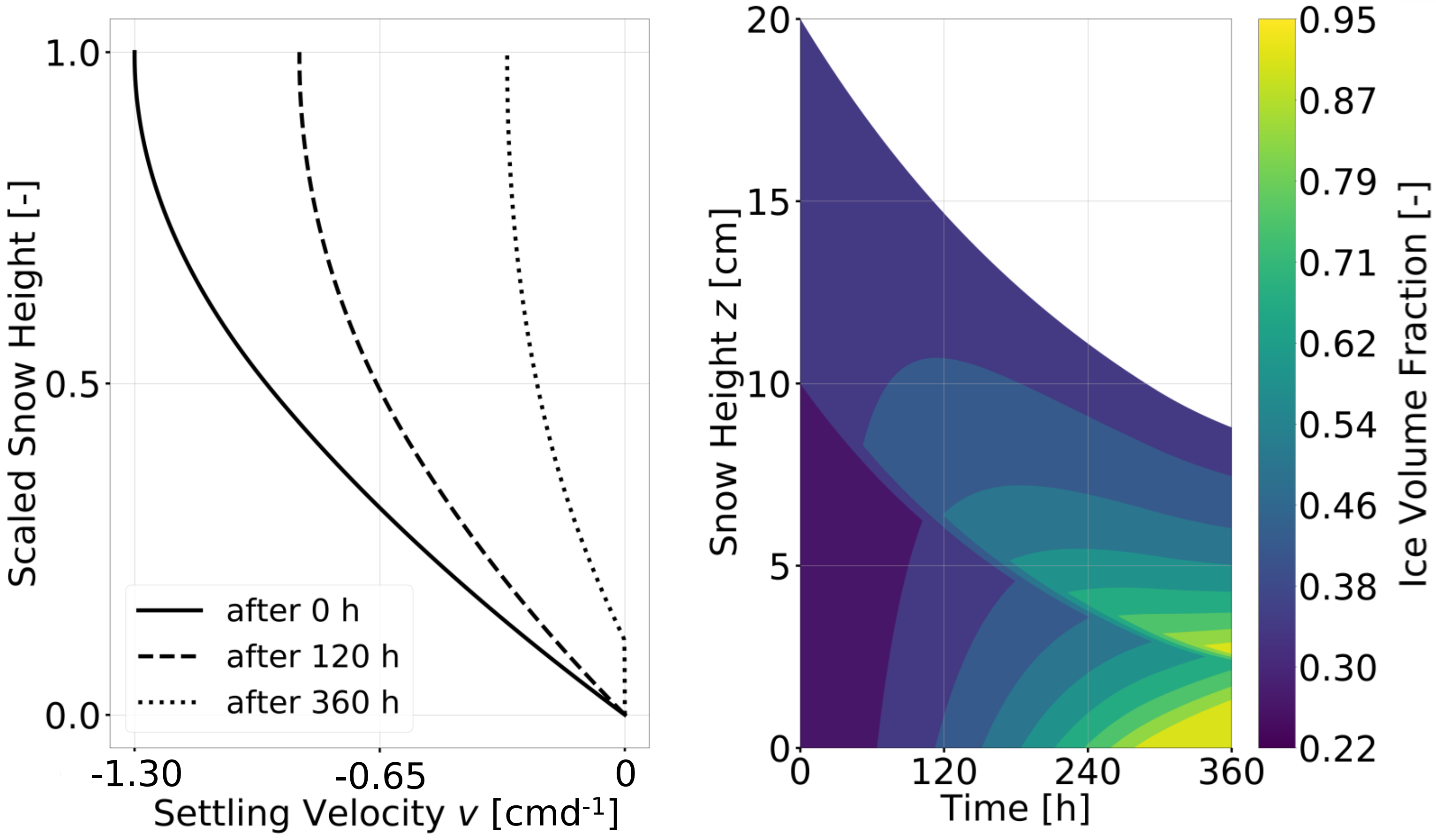
Settling Velocity – Stress controlled Strain Rate

Constant snow viscosity $\eta = 355211162\ Pas$[1]

$\eta \to \infty$ for $\phi > 0.95$ to constrain ice volume fraction to values lower than 1



We observe that:

- **Snow height**: decreases with realistic asymptote
- **Settling velocity:** non-linear, decreases with time
- **Layer thickness:** lower layer decreases faster
- **Ice volume fraction:** increases to maximum value 0.95

✓ Non-penetration
✓ Incompressibility
✓ Self-consolidation

[1] According to snow viscosity formulation from Vionnet et al. (2012) for T =263K and $\phi = 0.16$

Phase Change Term – Determined from a Conventional Process Model

Any type of continuum mechanical process model,
that allows for non-uniform grids can be coupled to MOC

Here, we test the coupling with the model from Hansen and Foslien (2015) [1]

Assumptions:

- Phase change covers water vapor and ice, so deposition and sublimation only

  Referred to as **condensation rate** c in the following: deposition $+c$ and sublimation $-c$

- Water vapor is always at saturation density
- Mechanical settling neglected

Mathematical model:

- Conservation equations for temperature, phase change, ice mass and energy

Results:

- Profiles for temperature and condensation rate

Extend mathematical model for our purposes: Add settling velocity

[1] More results of coupled computation can also be read in Simson (2019)

Adjust mathematical model for coupled phase change and settling

(1) Ice mass balance

$$\frac{\partial \phi}{\partial t} + \boldsymbol{\frac{\partial}{\partial z}(\phi \cdot v)} = \frac{c}{\rho_i}$$

(2) Water vapor mass balance [1]

$$(1-\phi)\cdot\frac{d\rho_v^{eq}}{dT}\frac{\partial T}{\partial t} - \frac{\partial}{\partial z}\left(D_{eff}\frac{d\rho_v^{eq}}{dT}\frac{\partial T}{\partial z}\right) \overbrace{+\,\boldsymbol{\rho_v^{eq}\cdot\frac{\partial}{\partial z}(\phi\cdot v)}}^{\ll 1} = -c$$

(3) Temperature equation

$$\left((\rho C)_{eff} + (1-\phi)\frac{d\rho_v^{eq}}{dT}\cdot L\right)\frac{\partial T}{\partial t} - \frac{\partial}{\partial z}\left(\left(k_{eff} + L\cdot D_{eff}\frac{d\rho_v^{eq}}{dT}\right)\frac{\partial T}{\partial z}\right) = \overbrace{\boldsymbol{-L\cdot\rho_v^{eq}\frac{\partial}{\partial z}(\phi\cdot v)}}^{\ll 1}$$

Computational workflow of the coupled system

Iterative process

Temperature	Condensation rate	Settling velocity	Ice volume fraction
Temperature equation (3)	Water vapor balance (2)	Stress-dependent	Ice mass balance (1)
Temperature gradient: 100 K/m			Initial condition: 2 Layer

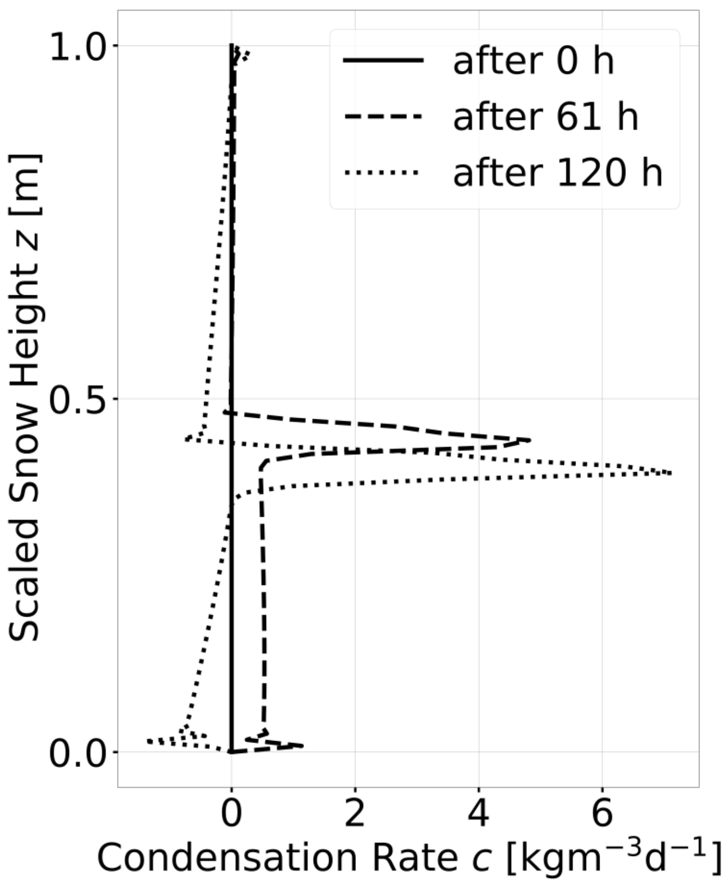
Simulation run for the coupled system

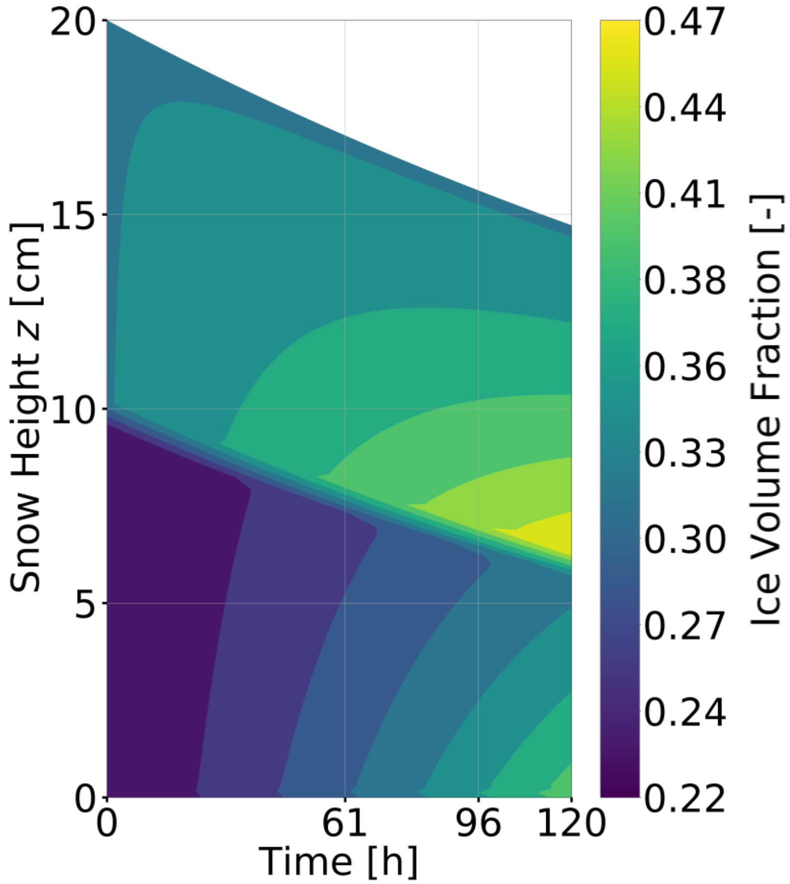
Legend

[1] Derivation in Appendix

Simulation Results – Settling Velocity Coupled to Process Model

Stress dependent settling and condensation rate





We observe that:

- **Snow height & layer thickness:** similar to settling only
- **Condensation rate:** Deposition in vicinity of layer boundary clearly visible

Open question:
Can competing effects be determined from ice volume fraction?

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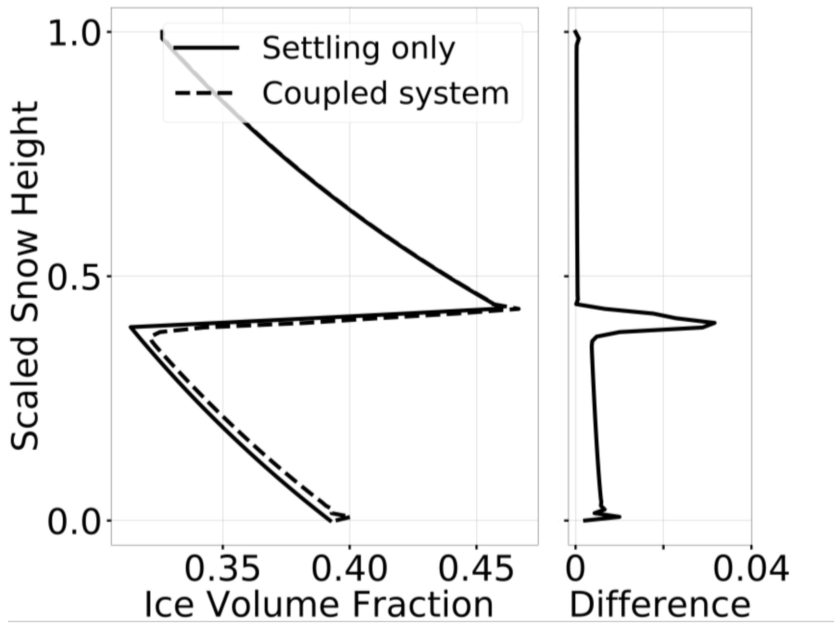
Simulation Results – Settling Velocity Coupled to Process Model

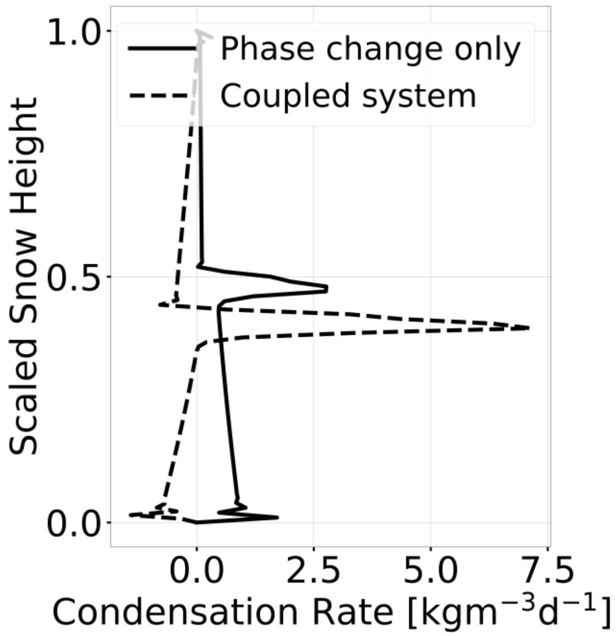
Assess competing effects after 5 days

Settling only vs. Coupled system



Coupling yields increase in ice volume fraction

Condensation rate only vs. Coupled system



Coupling yields increased deposition close to layer boundary

Conclusion and Outlook

The introduced model …

- is flexible: can be used with arbitrary strain rate formulations from snow and firn models
- is modular: competing effects of different processes can easily be tested
- combines advantages of Lagrangian and Eulerian formulations, e.g. preserving layer transitions in the ice phase while being coupled to Eulerian formulations of the vapor/water phase
- can be applied to arbitrary, continuous density profiles (does not rely on layers)

Future potential:

- Integrate further processes, e.g. evolution of specific surface area
- Use for model selection, to find dominant processes in common snow regimes

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Legend

ρ_i – Ice density $[\frac{kg}{m^3}]$ ϕ – Ice volume fraction $[-]$ c – Phase change rate $[\frac{kg}{m^3 \cdot s}]$ ρ_v^{eq} – Wator vapor density at saturation $[\frac{kg}{m^3}]$ D_{eff} – Effective diffusion coefficient $[\frac{s}{m^2}]$ z – Depth coordinates $[m]$ H – Total height $[m]$ L – Latent heat of ice $[\frac{J}{kg}]$	k_{eff} – Effective thermal conductivity $[\frac{W}{m \cdot K}]$ v – Settling velocity $[\frac{m}{s}]$ $(\rho C)_{eff}$ – Effective heat capacity $[\frac{J}{m^3 \cdot K}]$ $\dot{\varepsilon}$ – Strain rate $[s^{-1}]$ D_c – Strain rate coefficient $[s^{-1}]$ σ – Stress from overburdened mass $[Pa]$ η – Snow viscosity $[Pa \cdot s]$ t – Time $[s]$ T – Temperature $[K]$

Extended Mathematical Model – Water Vapor Mass Balance

Here $\frac{d\rho_v^{eq}}{dT}\frac{\partial T}{\partial t}$ is equivalent to $\partial_t \rho_v$ and $\frac{d\rho_v^{eq}}{dT}\frac{\partial T}{\partial z}$ is equivalent to $\partial_z \rho_v$

Ice mass balance: $\rho_i . \partial_t \phi + \rho_i \cdot \partial_z(\phi \cdot v) = c$

Water vapor mass balance: $(1-\phi) \cdot \partial_t \rho_v - \partial_z\left(D_{eff} \partial_z \rho_v\right) - \rho_v \partial_t \phi = -c$

$$(1-\phi) \cdot \partial_t \rho_v - \partial_z\left(D_{eff} \partial_z \rho_v\right) = -c + \rho_v \partial_t \phi \cdot \frac{\rho_i}{\rho_i}$$

Now, add terms on both sides to prepare substitution of second term on RHS

$$(1-\phi) \cdot \partial_t \rho_v - \partial_z\left(D_{eff} \partial_z \rho_v\right) + \boldsymbol{\rho_i \cdot \partial_z(\phi \cdot v) \cdot \frac{\rho_v}{\rho_i}} = -\boldsymbol{c} + \rho_i \partial_t \phi \cdot \frac{\rho_v}{\rho_i} + \boldsymbol{\rho_i \cdot \partial_z(\phi \cdot v) \cdot \frac{\rho_v}{\rho_i}}$$

Now, substitute second and third term on RHS with ice mass balance

$$(1-\phi) \cdot \partial_t \rho_v - \partial_z\left(D_{eff} \partial_z \rho_v\right) + \boldsymbol{\rho_v \cdot \partial_z(\phi \cdot v)} = \boldsymbol{c} \cdot \left(\frac{\rho_v}{\rho_i} - 1\right)$$

Assume $\frac{\rho_v}{\rho_i} \approx 0$

$$(1-\phi) \cdot \partial_t \rho_v - \partial_z\left(D_{eff} \partial_z \rho_v\right) + \boldsymbol{\rho_v \cdot \partial_z(\phi \cdot v)} = -\boldsymbol{c}$$

Legend