NP1.1. Mathematics of Planet Earth

EGU Mtg. Online (SGO)
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Climate-KIC

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The Lorenz Convection Model's Random Attractor (LORA) and its robust topology

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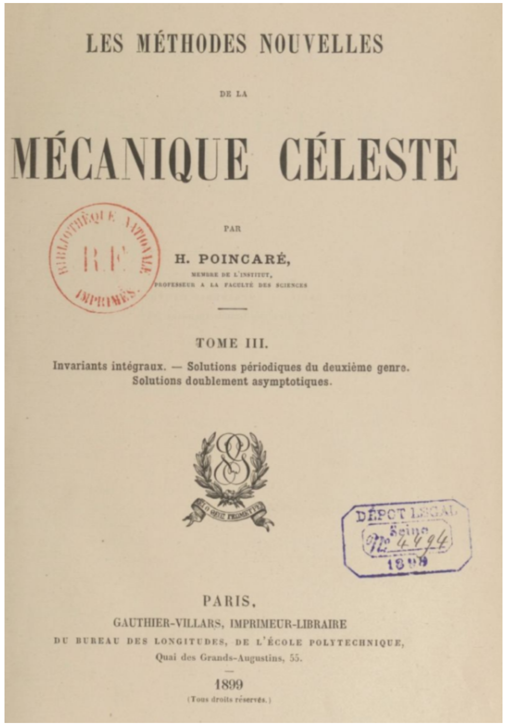
Please visit these sites for more info.

http://www.atmos.ucla.edu/tcd/, http://www.environnement.ens.fr/, and https://www.researchgate.net/profile/Michael_Ghil

H. Poincaré & the Topology of Chaos


LES MÉTHODES NOUVELLES

DE LA

MÉCANIQUE CÉLESTE

PAR

H. POINCARÉ,

MEMBRE DE L'INSTITUT,
PROFESSEUR A LA FACULTÉ DES SCIENCES

TOME III.

Invariants intégraux. — Solutions périodiques du deuxième genre.
Solutions doublement asymptotiques.

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
Quai des Grands-Augustins, 55.

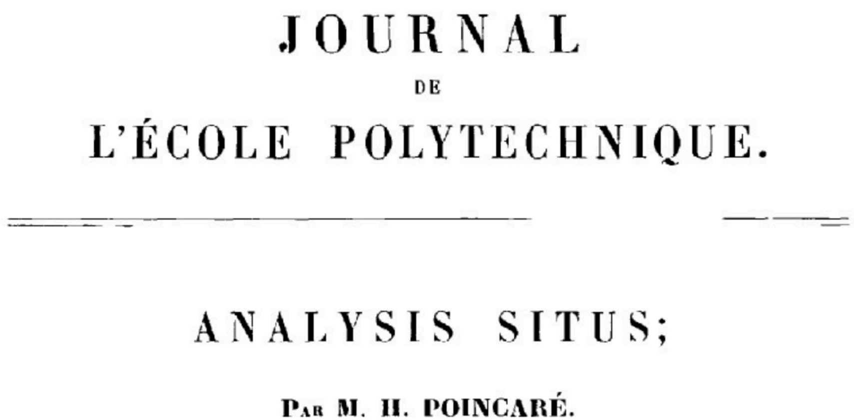
1899

(Tous droits réservés.)

Deterministic chaos theory starts with Henri Poincaré's study of the stability of the Solar System and the three-body problem (e.g., *Méthodes nouvelles de la Mécanique céleste*, vols. I–III, 1892–1899). He also clearly stated that weather was similarly affected by sensitive dependence on initial conditions (*Science et Méthode*, 1912, p. 69).

Poincaré also built on a few earlier results of Riemann, Betti & Jordan to create the field of algebraic topology, by introducing homology groups into what he called *analysis situs* (*J. Éc. Poly*., **1**, 1–121).

The work described herein will **combine these two strands** of Poincaré's heritage.


JOURNAL

DE

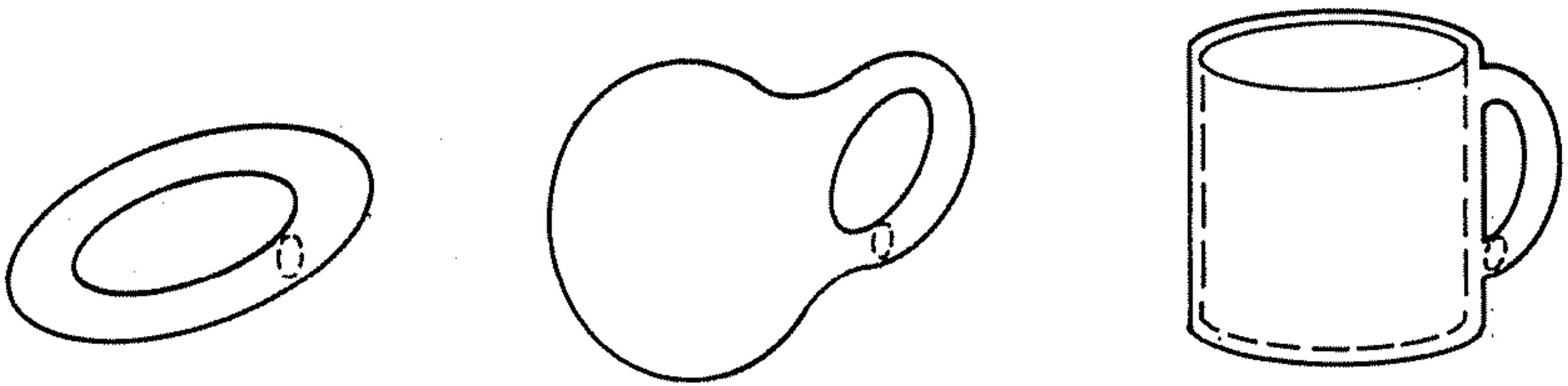
L'ÉCOLE POLYTECHNIQUE.

ANALYSIS SITUS;

PAR M. H. POINCARÉ.

What is topology?

- Topology studies the properties of objects that remain unaltered by continuous transformations.
- Detects properties that cannot be seen by geometry.
- Two objects are topologically equivalent if there is a continuous deformation from the one to the other (homeomorphism).

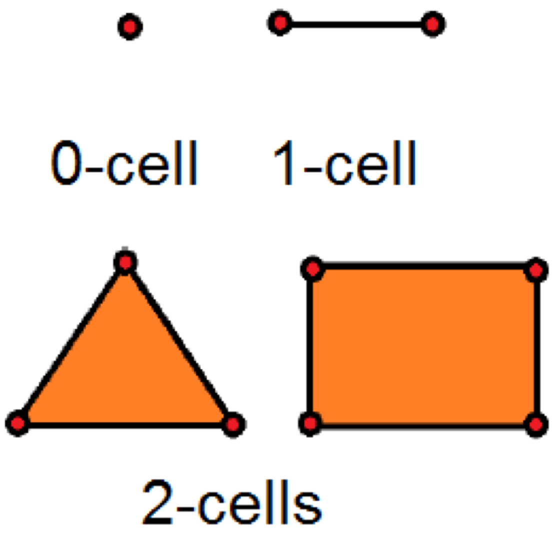


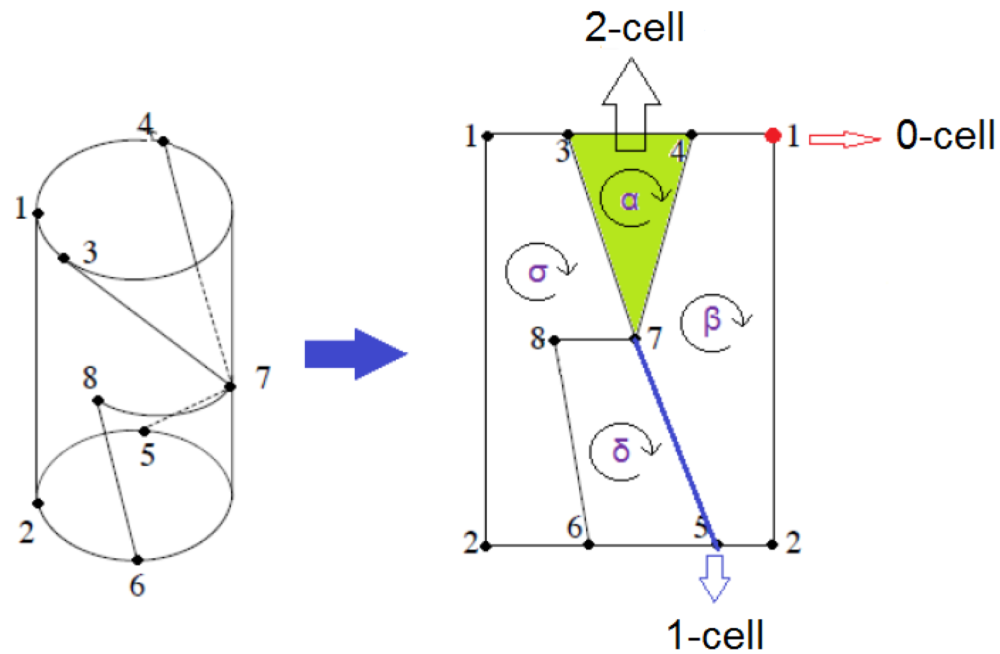
A donut and a coffee mug are topologically equivalent (Kinsey, 1993).

- It allows one to describe the connectivity of the object under study.

Algebraic topology – I

- Search for something computable to identify topological properties.
- Allows to turn a topological problem into an algebraic problem.
- Homology : seeks to make a distinction between topological spaces by determining the number of nonequivalent *n*-holes they contain. What is an *n*-hole?



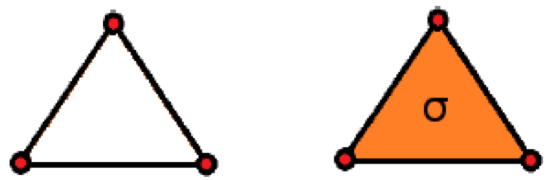


- Oriented cell complex: simplified structure ⇨ "Connectivity map of the cylinder"

Algebraic topology – II

Types of holes

- 0- holes — vertices
- 1- holes — cycles that are not the boundary of any 2-cells



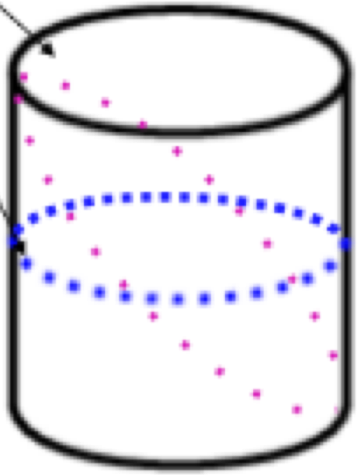
1 - hole? YES! NO!

- 2 - holes — cavities enclosed
- *N* - holes — hypercavities

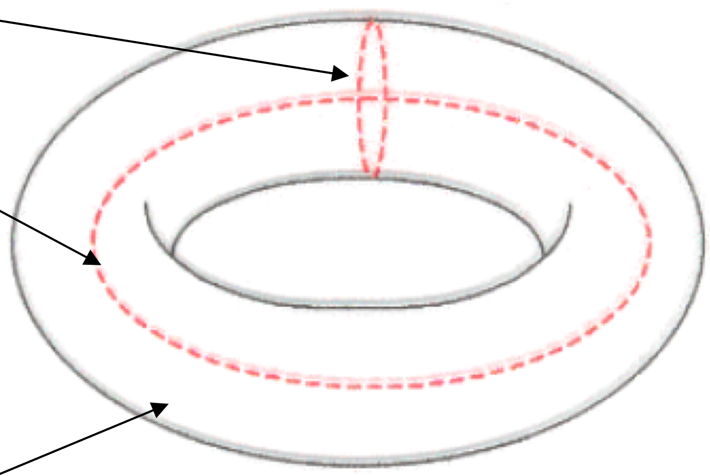
- Are we interested in all the *n*-holes? NO, we are interested only in the nonequivalent holes.
- Two holes are topologically equivalent if there is a smooth deformation from one hole to the other.

Algebraic topology – III

- 1- holes



one nonequiv. 1-hole



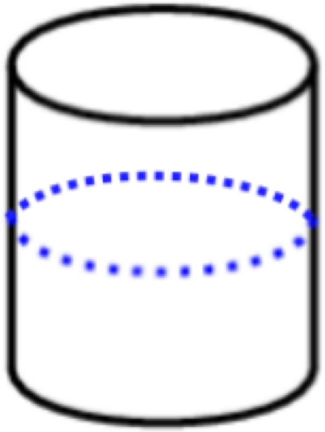
two nonequiv. 1-holes

- 2- holes

- How do we compute the nonequivalent n-holes? Recall that we are interested only in the nonequivalent holes.
- **Homology groups H_n**: identify the nonequivalent n-holes in the complex.
- Notation: k nonequivalent n-holes $H_n \sim \mathbb{Z}^k$

Algebraic topology – IV

Homology groups H_n: identify the k nonequivalent n-dimensional holes in the complex.



$H_0 \sim \mathbb{Z}$ one connected component

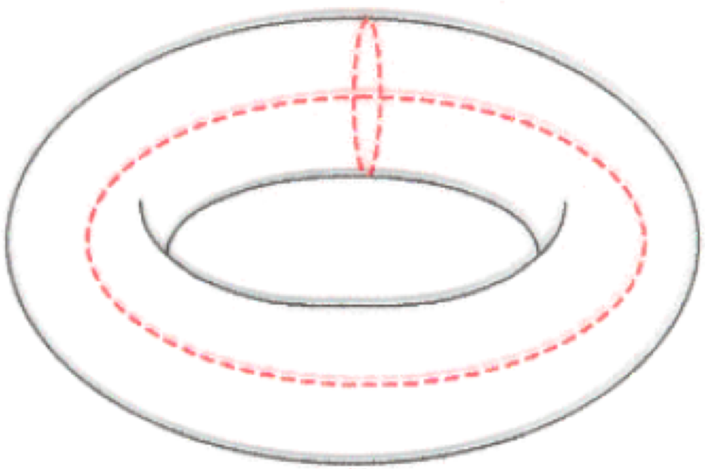
$H_1 \sim \mathbb{Z}$ one nonequiv. 1-D hole

$H_2 \sim \emptyset$ No cavities enclosed

one connected component $H_0 \sim \mathbb{Z}$

two nonequiv 1-D holes $H_1 \sim \mathbb{Z}^2$

one cavity enclosed $H_2 \sim \mathbb{Z}$

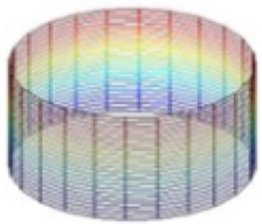


Algebraic topology – V

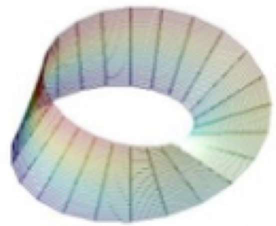
- How about a standard strip (cylinder) and a Moebius strip?

$$H_0 \sim \mathbb{Z}$$

$$H_1 \sim \mathbb{Z}$$



cylinder



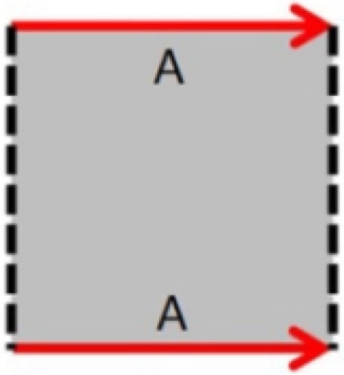
Moebius strip

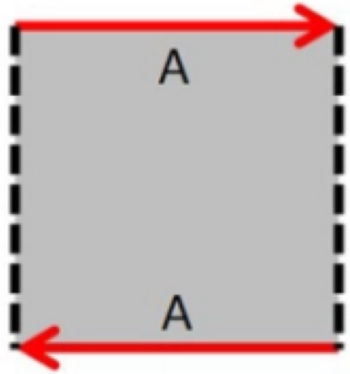
$$H_0 \sim \mathbb{Z}$$

$$H_1 \sim \mathbb{Z}$$

Torsion

cell complex





cell complex

No torsion

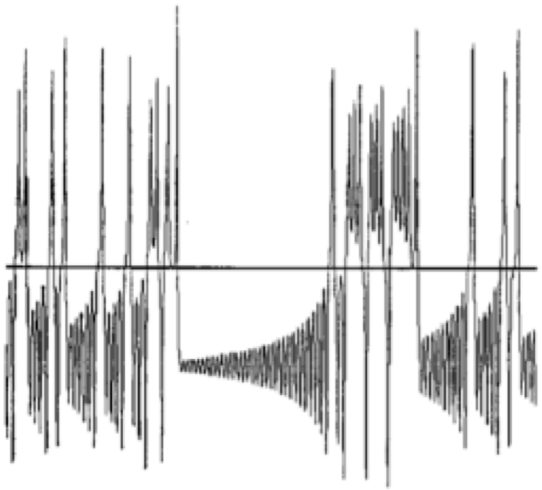
Torsion!

Orientability chains: allow one to identify torsions in an oriented cell complex.

Chaos topology – I

- It considers the problem of how *n*-dimensional trajectories or point-clouds are topologically structured in state space.

Relaxation oscillations in an optically pumped molecular laser (Gilmore,1998).



Holes!

Use topology to understand the flow dynamics.

Time series $x(t)$

Embedding projection onto a plane.

Simplified structure of the flow.

Reconstruction of the state space by
$(x(t), x(t-\tau), x(t-2\tau))$ delay embedding

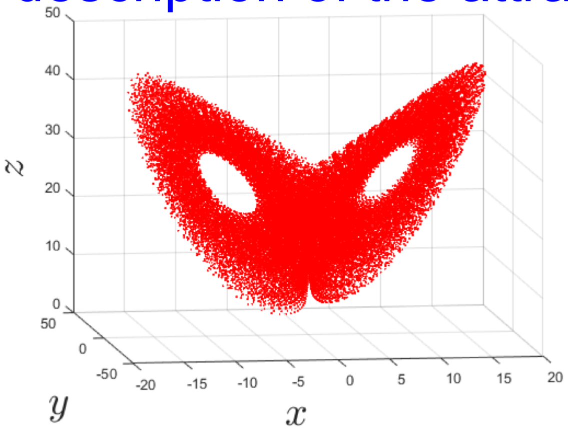
Branched manifold!

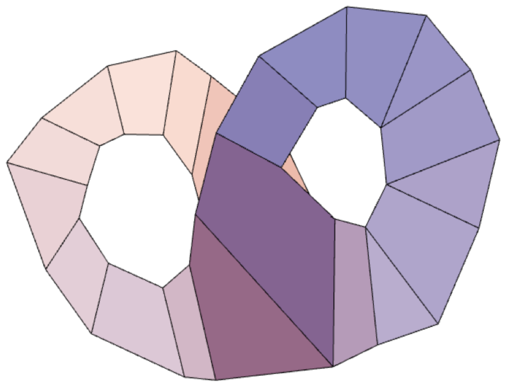
Chaos topology – II

- The dynamics on a deterministic attractor can be compactly described as the limit of a semi-flow on a **branched manifold (BM)**.
- The topological description of the BM encodes the invariant structure of the attractor in phase space.
- BM describes the topological organization of the trajectory on an attractor.

Topological description of the attractor ~ Topological description of the BM

Strange attractor





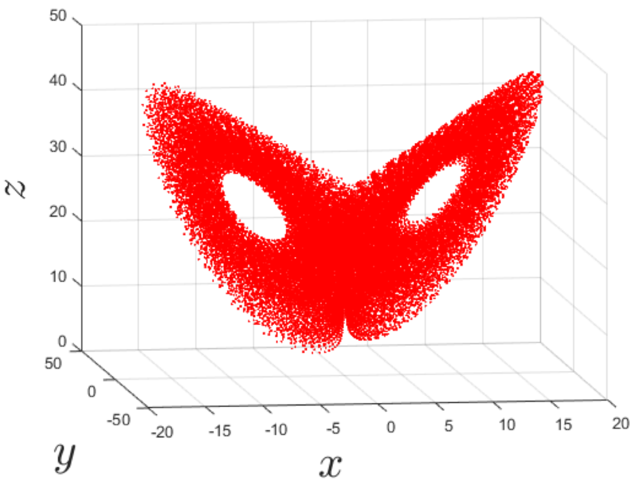
Branched manifold (BM)

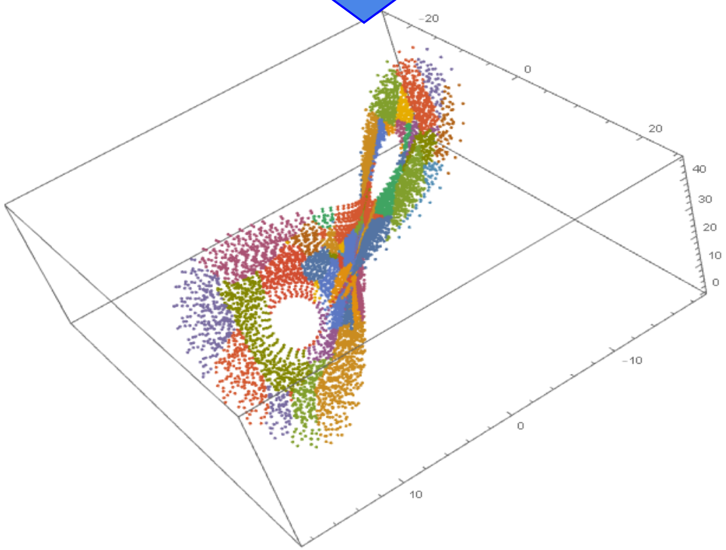
Reconstructing a BM from data amounts to:
(i) approximating a cloud of points in phase space by Euclidean closed sets;
(ii) forming a cell complex; and (iii) identifying the BM through the homology groups and torsions associated with the cell complex.

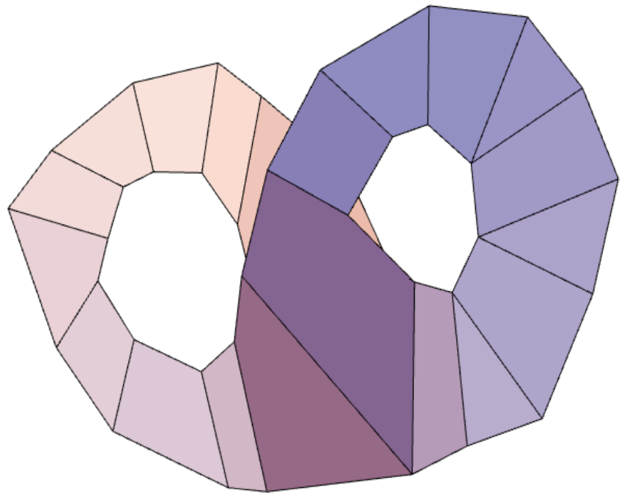
BraMAH method

BraMAH method: **Bra**nched **M**anifold **A**nalysis through **H**omologies
(Sciamarella & Mindlin,1999)

1. Decomposition into patches in $\mathbf{R}^n$. A ***patch*** is a set of points $\{\mathbf{x}_i\}$ around an arbitrary point $\mathbf{x}_0$ that is locally a linear approximation.
2. Construct a cell complex keeping track of the gluing prescriptions.
3. Compute homologies, and torsions of the cell complex.



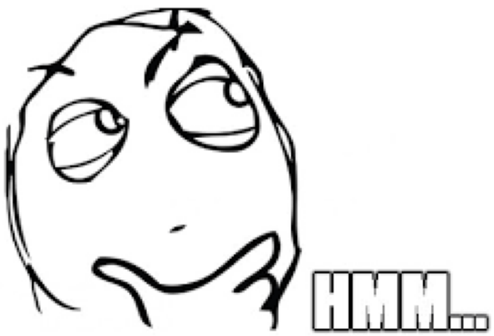




$$H_0 \sim \mathbb{Z}$$
$$H_1 \sim \mathbb{Z}^2$$
$$H_2 \sim \emptyset$$

Application: Random attractors

- And how about some “real stuff” now: chaotic + random?
- To address truly coupled climate–human behavior and climate change an important step is to examine time-dependent forcing.
- The proper framework for doing so is the theory of non-autonomous and random dynamical systems (NDS and RDS).
- Motivation: "A day in the life of the Lorenz (1963) model’s random attractor, or LORA for short; see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*) or Vimeo movie: https://vimeo.com/240039610."



LORA evolves in time. Does this evolution include changes in the topological sense?

LOrenz Random Attractor (LORA)

Stochastically perturbed Lorenz (1963) model.

$$\begin{aligned}
\mathrm{d}x &= s(y-x)\mathrm{d}t + \sigma x \mathrm{d}W_t, \\
\mathrm{d}y &= (rx - y - xz)\mathrm{d}t + \sigma y \mathrm{d}W_t, \\
\mathrm{d}z &= (-bz + xy)\mathrm{d}t + \sigma z \mathrm{d}W_t.
\end{aligned}$$

r = 28, *s* = 10, and *b* = 8/3, the usual values for the deterministically chaotic *strange attractor.*

In the system of nonlinear stochastic differential equations (SDEs) above, $W_t = W(t)$ is a Wiener process, and d*W* is Brownian motion. The noise is called multiplicative, because d*W* is multiplied by *x, y* or *z*.

Chekroun, Simonnet & Ghil (*Physica D*, 2011)

LORA

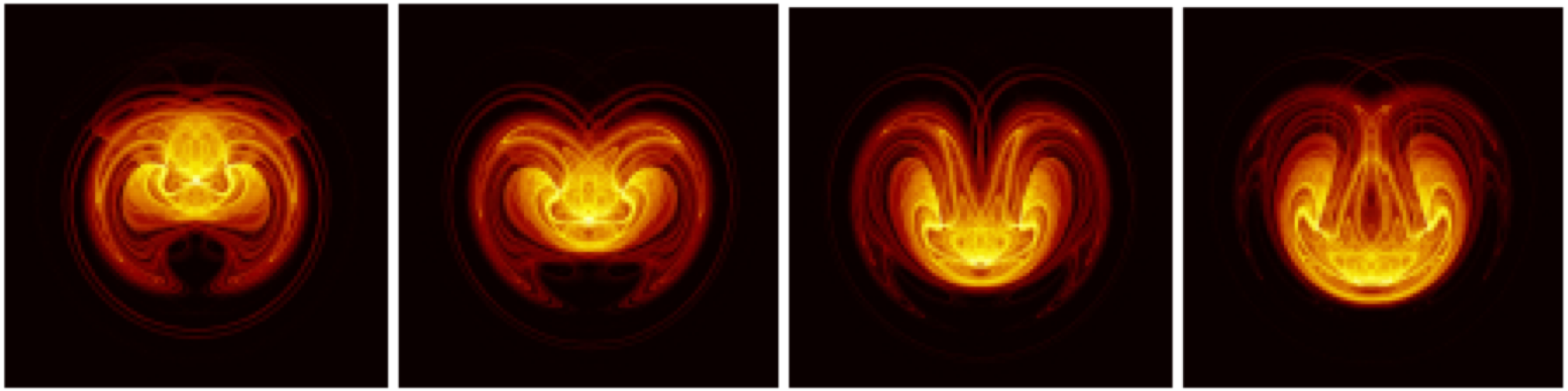
LORA is a random attractor in $\boldsymbol{R}^3$

- The point cloud is obtained for a fixed realization ω of the noise, and it is associated with a particular time instant, $t = t_0$.
- To construct a random attractor a lot of initial particles are used.
- Each point within the cloud can be associated to a μ-value, with low μ-values corresponding to the less populated regions of the random attractor, i.e., lower probability density function (pdf).

LORA snapshot. The figure corresponds to projection onto the (y,z)-plane of the sample measure μ (in yellow), cf. Chekroun et al. (*Physica D*, 2011).
One billion of initial particles are used.

The topology of LORA – I

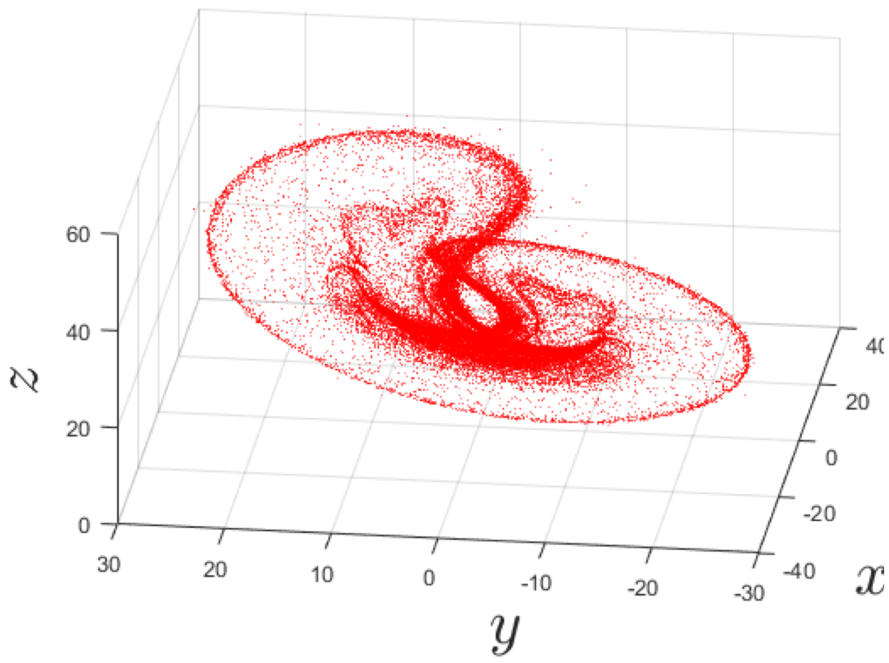
- While the attractor associated with the classical, deterministic Lorenz (1963) model is “strange" but fixed in time, the Lorenz model’s attractor (LORA), driven by multiplicative noise, is a **random attractor** that changes in time.



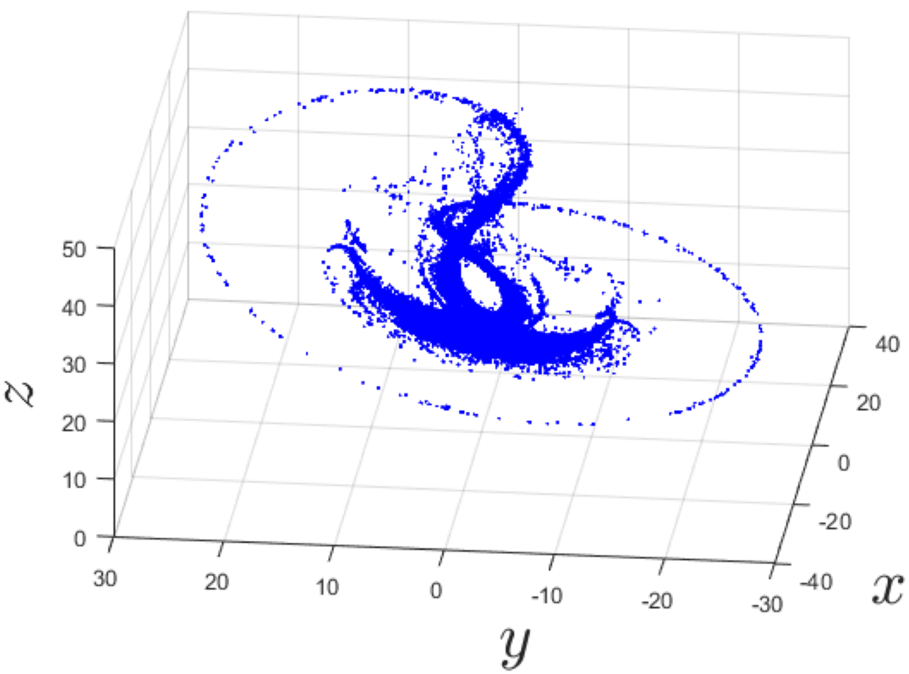
- We describe the temporal evolution of the topological structure of the branched manifold (BM) associated with LORA.
- In order to study the changes in LORA's topological structure, BraMAH is extended from deterministically chaotic flows to nonlinear noise-driven systems.

The topology of LORA – II

We are interested in characterizing the topology of the most populated regions of the point cloud, and in order to do this, we select a threshold in the μ-value. This value is chosen in order to guarantee the convergence of the point cloud's topology.

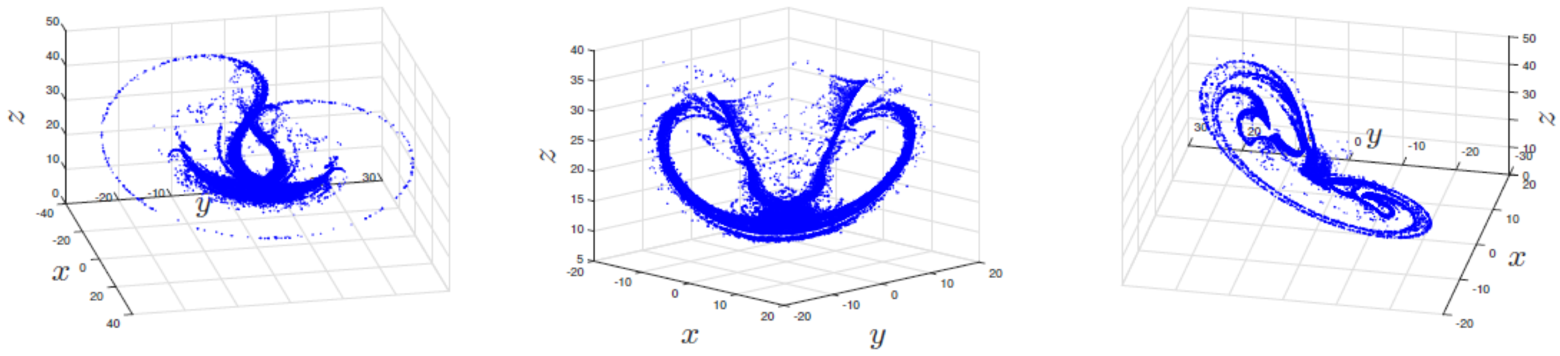


LORA at t = 40.09 (red),
10 000 initial particles

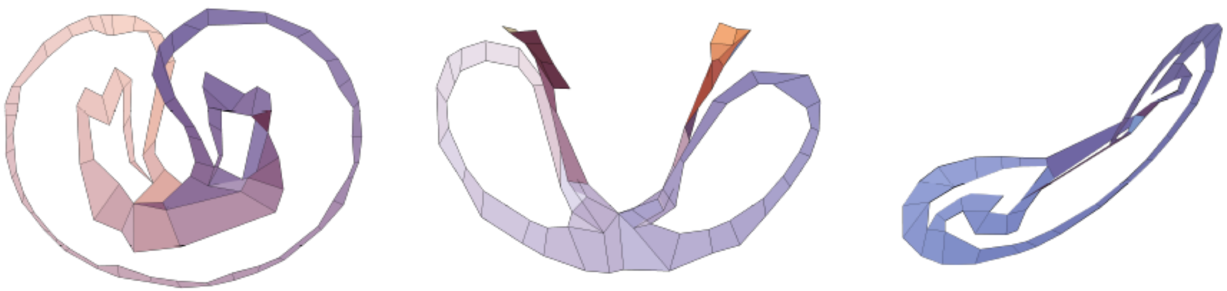


The point cloud after the application of the threshold (blue)

The topology of LORA – III



$t = 40.09$ $t = 40.18$ $t = 40.27$



$H_1 \sim \mathbb{Z}^3$ $H_1 \sim \mathbb{Z}^2$ $H_1 \sim \mathbb{Z}^4$

The topology of LORA – IV

Conclusion

LORA is a 2-dimensional branched manifold whose **number of 1-D holes changes over time**, i.e., its homology group H_1 is distinct from the fixed one of Lorenz's "butterfly" and **cycles are created or destroyed by the noise**.

Work in progress

- When the amount of initial points is increased, small features (more holes) could appear. The definition of the structure is increased to an "HD LORA." We are studying how this finer structure is captured by other "paths" in the attractor, yielding a "sub-branched manifold".

- We study changes of the homology groups as the variance σ of the noise is modified.

Bibliography

- **Chekroun, M. D.,** E. Simonnet, and M. Ghil, 2011: "Stochastic climate dynamics: Random attractors and time-dependent invariant measures." *Physica D: Nonlinear Phenomena* 240(2): 1685-1700.
- **Crauel, H.** and F. Flandoli, 1994: Attractors for random dynamical systems. *Probability Theory and Related Fields.* 100(3), pp.365-393.
- **Ghil, M.**, 2019: A century of nonlinearity in the geosciences, *Earth & Space Science*, **6**, 1007–1042, doi: 10.1029/2019EA000599.
- **Ghil, M.,** and V. Lucarini, 2019: The physics of climate variability and climate change, *Reviews of Modern Physics,* in press, arXiv:1910.00583.
- **Gilmore, R.**, 1998: Topological analysis of chaotic dynamical systems. *Reviews of Modern Physics* 70(4): 1455.
- **Kinsey, L. C.,** 1997: *Topology of Surfaces*. Springer Science & Business Media.
- **Lorenz, E. N.,** 1963: Deterministic nonperiodic flow. *J. Atmospheric Sciences* 20(2): 130–141.
- **Lefranc, M.,** 2002: *The Topology of Chaos: Alice in Stretch and Squeezeland*. Wiley-Interscience.
- **Sciamarella, D.,** and G. B. Mindlin, 1999:. *Physical Review Letters* 82(7): 1450. Topological structure of chaotic flows from human speech from data
- **Sciamarella, D.,** and G. B. Mindlin, 2001: Unveiling the topological structure of chaotic flows from data. *Physical Review E* 64(3): 036209 (pp. 1–9).