# Troposphere-Stratosphere Coupling and Resonant Over-Reflection of Rossby Waves: The Role of Critical Layer Nonlinearity

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## Motivation

There exists a **coupling** mechanism between the troposphere and the stratosphere, which plays a fundamental role in weather and climate. The coupling is highly complex and rests upon radiative and chemical feedbacks, as well as dynamical coupling by Rossby waves:

- The troposphere acts as a source of Rossby waves which propagate **upwards** in to the stratosphere, affecting the zonal mean flow.
- Rossby waves are also likely to play a significant role in downward communication of information. This happens via **downward reflection** from the stratosphere in to the troposphere.
- Shaw and Perlwitz (2013) analysed the two-way coupling via wave reflection and quantified the impact of the downward wave coupling mechanism on the troposphere, suggesting that wave reflection can directly influence tropospheric weather.
- A shear flow exhibits a so-called critical layer, where viscous and nonlinear effects become important.
- A wave incident upon a critical layer may be absorbed, reflected or overreflected, whereby the amplitude of the reflected wave is larger than that of the incident wave. The concept of critical layer overreflection is key to understanding atmospheric instability.

The ultimate aim is to prove a self-consistent mathematical framework on this coupling.

## Formulation

- A shear flow $U(y)$ is considered, and small perturbations imposed. The total flow field can be expressed as:

$$\psi = \int U \, \mathrm{d}y + \epsilon \tilde{\psi}(x, y, t).$$

- In the main part of the shear flow, viscosity and nonlinear effects are negligible when $Re \gg 1$ and $\epsilon \ll 1$. This is considered to be the 'Outer Region'.
- The streamfunction is sought as a superposition of normal modes

$$\tilde{\psi} = \Re\left\{ \sum_{n=1}^{\infty} \hat{\psi}^{(n)}(y) e^{in(\alpha x - \omega t)} \right\},$$

and the governing equation is the Rayleigh-Kuo equation

$$(U - c)\left( \frac{d^2 \hat{\psi}^{(n)}}{dy^2} - n^2 \alpha^2 \hat{\psi}^{(n)} \right) + \left( \beta - \frac{d^2 U}{dy^2} \right) \hat{\psi}^{(n)} = 0.$$

- This exhibits a singularity where $U = c$ at $y = y_c$, referred to as the **critical line**. The solutions to the Rayleigh-Kuo equation close to the singularity are the Tollmien (1929) solutions, $\phi_1$ and $\phi_2$:

$$\hat{\psi}^{(n)}(y) = A_{\pm}^{(n)} \hat{\phi}_1^{(n)}(y_c \pm \delta) + B^{(n)} \hat{\phi}_2^{(n)}(y_c \pm \delta).$$

- This suggests the existence of a **critical layer** close to y = yc, where extra effects such as viscosity and/or nonlinearity must be introduced. Different expansions must then be constructed and a different solution sought, known as the inner solution.
- Traditionally, only the fundamental harmonics ($n = 1$) has been considered, under the assumption that the effect of the higher harmonics is negligible. However, it has been shown that this is not the case, and nonlinearly generated harmonics spread out to the entire shear flow.
- The parameter $\lambda$ can be introduced as

$$\lambda = \frac{1}{Re} \epsilon^{-\frac{3}{2}}.$$

$\lambda$ describes the relative importance of nonlinearity against viscosity in the critical layer. As $\lambda \to 0$, nonlinearity dominates in the critical layer, and as $\lambda \to \infty$, viscosity dominates in the critical layer.

## The Domain

$$\hat{\psi}^{(n)}(y) \to Re^{ik_+^{(n)} y} + e^{ik_-^{(n)} y} \quad \text{as } y \to \infty, \quad k_{\pm}^{(n)} = \pm\sqrt{\frac{\beta}{1-c} - n^2\alpha^2}$$

**Outer Region**

$$\hat{\psi}^{(n)}(y) = A_+^{(n)} \hat{\phi}_1^{(n)}(y_c + \delta) + B^{(n)} \hat{\phi}_2^{(n)}(y_c + \delta)$$

**Critical Layer** $y = y_c$

$$\hat{\psi}^{(n)}(y) = A_-^{(n)} \hat{\phi}_1^{(n)}(y_c - \delta) + B^{(n)} \hat{\phi}_2^{(n)}(y_c - \delta)$$

$$\hat{\psi}^{(n)}(y) \sim e^{il_+^{(n)} y} \quad \text{as } y \to -\infty, \quad l_{\pm}^{(n)} = \pm\sqrt{n^2\alpha^2 + \frac{\beta}{1+c}}$$

## Jump Across The Critical Layer

Jump conditions can be derived across the critical layer. In particular, for the fundamental harmonic $n = 1$, the jump condition is:

$$J_1 = \frac{U_c'}{2(U_c'' - \beta)B^{(1)}} (A_+^{(1)} - A_-^{(1)}).$$

$J_1$ varies monotonically between 0 and $\pi/2$ depending on whether nonlinearity or viscosity dominates. When harmonics are considered within the critical layer but assumed to be negligible in the outer region, the following results are obtained:

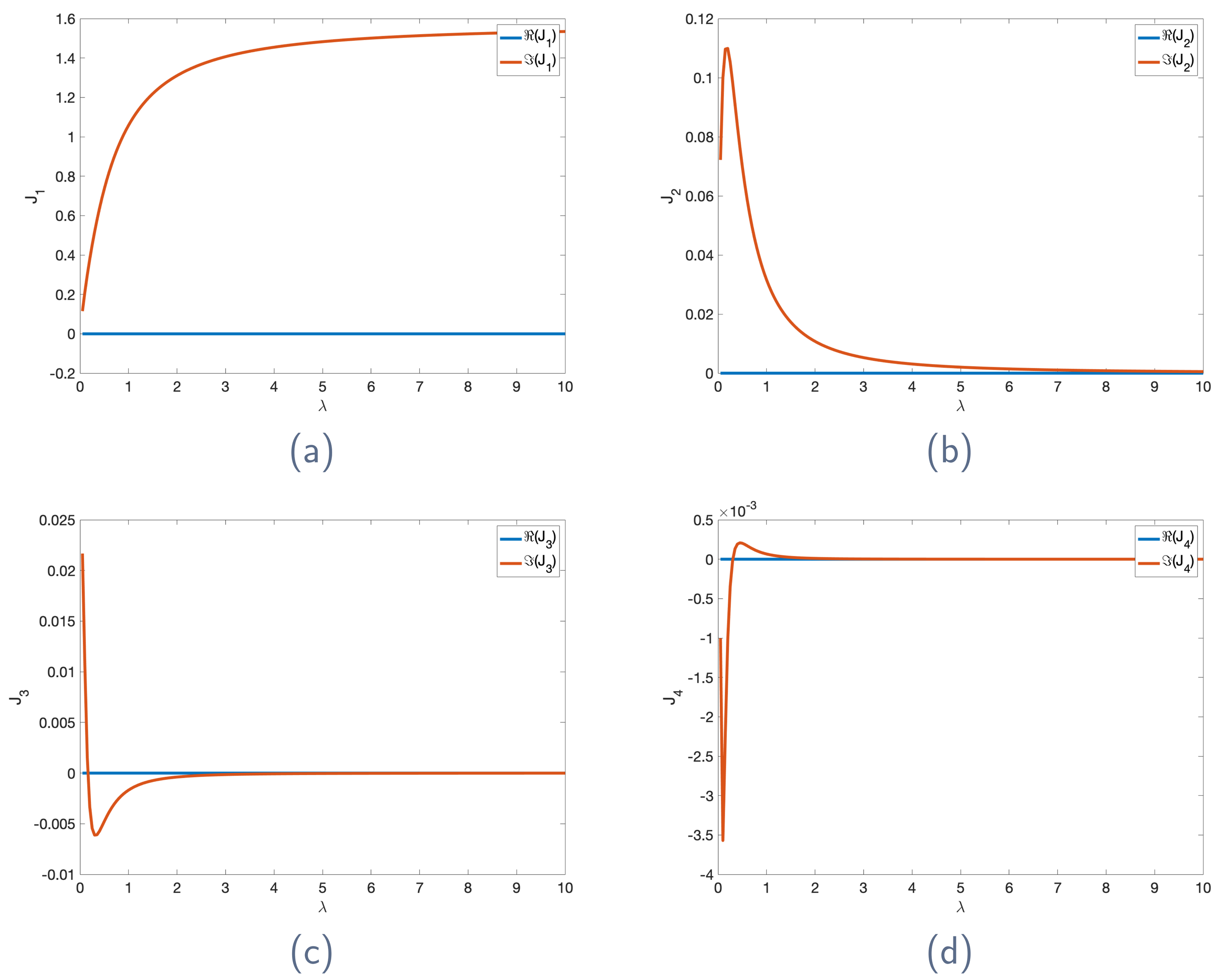
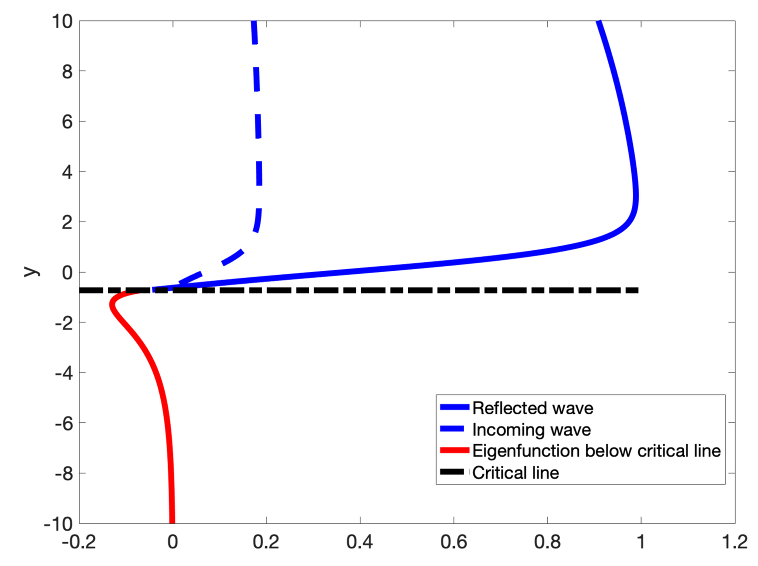


Figure: Jump across critical layer for (a) fundamental harmonic, $J_1$, (b) second harmonic, $J_2$, (c) third harmonic, $J_3$ and (d) fourth harmonic, $J_4$.

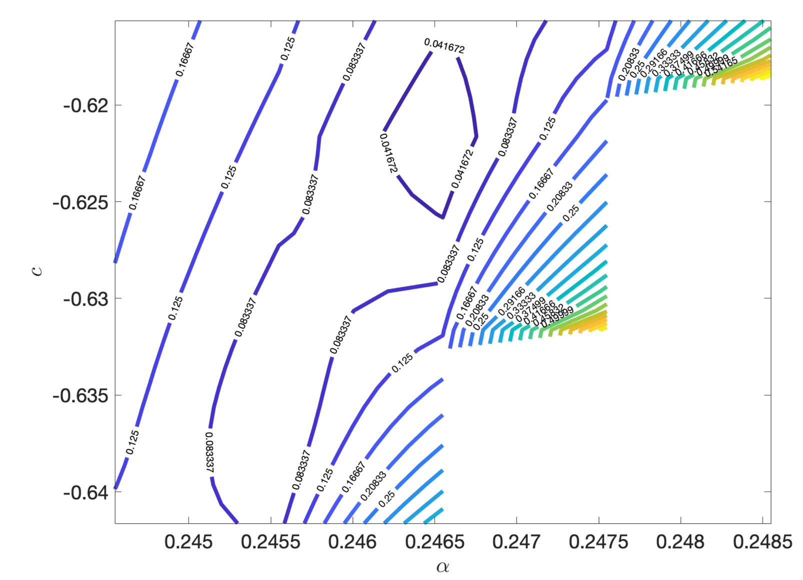
This is significant: it implies that the higher harmonics spread out to the entire shear flow. In previous analyses they had been ignored because their effect was assumed to be small, and so a reformulation of classical problems is needed.

## Reflection of Rossby Waves

- The reflection of Rossby waves by a rotating zonal shear layer was investigated in terms of a viscous critical layer in the stratosphere.
- Observational evidence for these critical surfaces have been demonstrated by Perlwitz & Harnik (2003, 2004) and Song & Robinson (2004).
- A special case of wave overreflection is "resonant overreflection", where the shear layer appears to spontaneously emit an outgoing wave without the presence of an incident wave. This corresponds to an infinite reflection coefficient, and appropriate physical constraints must be considered and introduced in order to regularise the problem.
- Numerical evidence has been found of resonant overreflection for the velocity profile $U(y) = \tanh(y)$ on an infinite domain. For $\beta = 0.1$, an infinite reflection coefficient is found at $(\alpha, c) = (0.2465488, -0.6216327)$. This corresponds to the so-called resonant over-reflection, and but it also signifies the existence of a radiating Rossby wave.




(a) Visualisation of $\hat{\psi}$, where the upper far-field condition is that of a transmitted and reflected wave, and the lower far-field condition is that of exponential decay.



(b) A contour plot of $1/|R|$ against $c$ and $\alpha$ for $\beta = 0.1$. The root is found at $(\alpha, c) = (0.2465488, -0.6216327)$.

**Further work:**

- Only a viscous critical layer has been studied - an investigation of the effects of nonlinearity is required in order to determine the full-effects of the higher harmonics.
- So far only an idealistic velocity profile has been considered - ultimately, we will consider a much more realistic profile, seen in a baroclinic setting.
- The physical constraints of resonant overreflection need to be considered, and the problem must be regularised to establish a relationship between the incident and reflected waves.

## References

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