LINEAR AND NONLINEAR WAVES IN THREE-DIMENSIONAL STRATIFIED ROTATING ASTROPHYSICAL FLOWS IN THE BOUSSINESQ APPROXIMATION

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EGU General Assembly ST1.9 Sharing Geoscience Online

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INTRODUCTION

- We consider a stably-stratified layer of rotating plasma in gravitational field in Boussinesq approximation with linear density profile
- Taking into account effects of stratification in magnetohydrodynamic models of a rotating plasma is important for the analysis of many phenomena and astrophysical objects, such as
  - solar tachocline
  - stably-stratified regions in stars (radiating zones) and planets (outer liquid layer of the core)
  - oscillations of rotating stars and the Sun
  - astrophysical disks and exoplanets
- Moreover, it allows one to significantly expand the possibilities for interpreting the available observational data of large-scale Rossby waves on the Sun

TRADITIONAL AND NON-TRADITIONAL f-PLANE APPROXIMATION

$$
\begin{cases}
\dfrac{\partial \vec{u}}{\partial t} + \vec{f} \times \vec{u} + \nabla P + \rho' \hat{\vec{z}} + \overrightarrow{B_0} \times \left(\nabla \times \vec{B}\right) = 0, & \text{linearized momentum equation} \\
\dfrac{\partial \vec{B}}{\partial t} - (\vec{B}_0 \nabla)\vec{u} = 0, & \text{linearized equation for magnetic field} \\
\dfrac{\partial \rho'}{\partial t} + N^2 u_z = 0, & \text{linearized equation for density} \\
\operatorname{div} \vec{u} = 0. & \text{linearized divergence-free condition for velocity}
\end{cases}
$$

Background state:

$$\overrightarrow{u_0} = 0, \overrightarrow{B_0} = const,$$

$$\frac{\partial P_0}{\partial z} = -\bar{\rho}(z),\ \bar{\rho}(z) = N^2 z \frac{\widetilde{\rho_0}}{g}$$

N^2 – Brunt–Väisälä frequency

$$\rho' = \frac{\rho g}{\widetilde{\rho_0}},\quad P = \frac{p}{\widetilde{\rho_0}},\quad \vec{B} = \frac{\vec{b}}{\sqrt{4\pi\widetilde{\rho_0}}}$$

In f-plane approximation we suppose that Coriolis parameter $\vec{f} = 2\vec{\Omega}$ is constant

traditional

$$\vec{f} = (0,0,f_V),\ f_V = 2\Omega \sin\theta$$

non-traditional

$$\vec{f} = (0, f_H, f_V),\ f_V = 2\Omega \sin\theta,\ f_H = 2\Omega \cos\theta$$

taking into account the horizontal component of the Coriolis force

SOLUTION OF DISPERSION EQUATION ON TRADITIONAL f-PLANE IN FORM OF MAGNETIC INERTIA-GRAVITY WAVES

$$\omega^4 - \omega^2\left(f_V^2\frac{k_z^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) + (B_0\cdot k)^2\left((B_0\cdot k)^2 - N^2\frac{k_h^2}{k^2}\right) = 0,\quad \boldsymbol{k_h} = (k_x, k_y, 0)$$

$$\boldsymbol{\omega_{mig_{3D}} = \pm\sqrt{\frac{1}{2}\left(f_V^2\frac{k_z^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) + \frac{1}{2k^2}\sqrt{f_V^4 k_z^4 + 4(B_0\cdot k)^4 f_V^2\frac{k_z^2}{k^2} - 2f_V^2k_z^2N^2k_h^2 + N^4k_h^4}}} \quad \mathbf{(1)}$$

$\boxed{\boldsymbol{B_0 = 0}} \Longrightarrow \omega_{gr_{3D}} = \pm\sqrt{f_V^2\frac{k_z^2}{k^2} - N^2\frac{k_h^2}{k^2}}$

In the absence of magnetic field dispersion relation (1) describes inertio-gravity wave in neutral fluid, for which is satisfied the condition of group velocity perpendicularity to wave vector $\boldsymbol{v}_{ig_{3D}}\cdot\boldsymbol{k} = 0$. We find out that in the presence of magnetic field this condition is violated: $\boldsymbol{v}_{mig_{3D}}\cdot\boldsymbol{k} \neq 0$

$\boxed{k_z \ll \boldsymbol{k}} \Longrightarrow \omega_a = \pm(B_0\cdot k)_h$

In the case of waves propagation in horizontal plane (k_x, k_y) dispersion relation (1) describes Alfven waves

$\boxed{\boldsymbol{k} = k_z} \Longrightarrow \omega_{z_1} = \pm\sqrt{\frac{f_V^2}{2} + B_{0_z}^2k_z^2 + f_V\sqrt{\frac{f_V^2}{4} + B_{0_z}^2k_z^2}}$ (2)

In the case of waves propagation along the vertical component of wave vector dispersion relation (1) describes vertical magnetic waves with frequency ω_{z_1} (2)

SOLUTION OF DISPERSION EQUATION ON TRADITIONAL f-PLANE IN FORM OF MAGNETOSTROPHIC WAVES

$$\omega^4 - \omega^2\left(f_V^2\frac{k_z^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) + (B_0\cdot k)^2\left((B_0\cdot k)^2 - N^2\frac{k_h^2}{k^2}\right) = 0,\quad \boldsymbol{k_h} = (k_x, k_y, 0)$$

$$\boldsymbol{\omega_{mstr_{3D}}} = \pm\sqrt{\frac{1}{2}\left(f_V^2\frac{k_z^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) - \frac{1}{2k^2}\sqrt{f_V^4k_z^4 + 4(B_0\cdot k)^4 f_V^2\frac{k_z^2}{k^2} - 2f_V^2k_z^2N^2k_h^2 + N^4k_h^4}} \qquad \textbf{(3)}$$

$\boldsymbol{B_0 = 0}$ $\Longrightarrow$ none

This type of wave has **no analogue** in neutral fluid dynamics and disappears in the absence of magnetic field

$k_z \ll \boldsymbol{k}$ $\Longrightarrow$ $\omega_{mgr} = \pm\sqrt{(B_0\cdot k)_h^2 - N^2}$

In the case of waves propagation in horizontal plane (k_x, k_y) dispersion relation (3) describes magnetic gravity waves

$\boldsymbol{k} = k_z$ $\Longrightarrow \omega_{z_2} = \pm\sqrt{\frac{f_V^2}{2} + B_{0_z}^2k_z^2 - f_V\sqrt{\frac{f_V^2}{4} + B_{0_z}^2k_z^2}}$ (4)

In the case of waves propagation along the vertical component of wave vector dispersion relation (3) describes vertical magnetic waves with frequency ω_{z_2} (4)

SOLUTION OF DISPERSION EQUATION ON NON-TRADITIONAL f-PLANE IN FORM OF MAGNETIC INERTIO-GRAVITY WAVES

$$\omega^4 - \omega^2\left(\frac{\left(f_H k_y + f_v k_z\right)^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) + (B_0\cdot k)^2\left((B_0\cdot k)^2 - N^2\frac{k_h^2}{k^2}\right) = 0, \qquad \boldsymbol{k_h} = (k_x, k_y, 0)$$

$$\boldsymbol{\omega_{mig'_{3D}}} = \pm\sqrt{\frac{1}{2}\left(\frac{\left(f_H k_y + f_v k_z\right)^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) + \frac{1}{2k^2}\sqrt{\left(f_H k_y + f_v k_z\right)^4 + 4(B_0\cdot k)^4\frac{\left(f_H k_y + f_v k_z\right)^2}{k^2} - 2\left(f_H k_y + f_v k_z\right)^2 N^2 k_h^2 + N^4 k_h^4}} \quad \textbf{(5)}$$

$\boldsymbol{B_0 = 0}$ $\Longrightarrow$ $$\omega_{gr'_{3D}} = \pm\sqrt{\frac{\left(f_H k_y + f_v k_z\right)^2}{k^2} - N^2\frac{k_h^2}{k^2}}$$

In the absence of magnetic field dispersion relation (5) describes inertio-gravity wave in neutral fluid

$k_z \ll \boldsymbol{k}$ $\Longrightarrow$ $$\omega_{mig'} = \pm\sqrt{\frac{f_H^2 k_y^2}{2k_h^2} - \frac{N^2}{2} + (B_0\cdot k)_h^2 + \sqrt{\frac{f_H^2 k_y^2}{4k_h^2}\left(\frac{f_H^2 k_y^2}{k_h^2} - 2N^2 + 4(B_0\cdot k)_h^2\right) + \frac{N^4}{4}}} \quad (6)$$

in the case of waves propagation in horizontal plane (k_x, k_y) dispersion relation (5) describes magnetic inertio-gravity waves (6)

In the case of waves propagation along the vertical component of wave vector dispersion relation (5) describes vertical magnetic waves with frequency ω_{z_1}(2)

SOLUTION OF DISPERSION EQUATION ON NON-TRADITIONAL f-PLANE IN FORM OF MAGNETOSTROPHIC WAVES

$$\omega^4 - \omega^2\left(\frac{(f_H k_y + f_v k_z)^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) + (B_0\cdot k)^2\left((B_0\cdot k)^2 - N^2\frac{k_h^2}{k^2}\right) = 0, \qquad \boldsymbol{k_h} = (k_x, k_y, 0)$$

$$\boldsymbol{\omega_{mstr'_{3D}} = \pm\sqrt{\frac{1}{2}\left(\frac{(f_H k_y + f_v k_z)^2}{k^2} - N^2\frac{k_h^2}{k^2} + 2(B_0\cdot k)^2\right) - \frac{1}{2k^2}\sqrt{(f_H k_y + f_v k_z)^4 + 4(B_0\cdot k)^4\frac{(f_H k_y + f_v k_z)^2}{k^2} - 2(f_H k_y + f_v k_z)^2 N^2 k_h^2 + N^4 k_h^4}} \quad (7)$$

$\boxed{\boldsymbol{B_0 = 0}}$ $\Longrightarrow$ $\boxed{\text{none}}$

This type of wave has **no analogue** in neutral fluid dynamics and disappears in the absence of magnetic field

$\boxed{k_z \ll \boldsymbol{k}}$ $\Longrightarrow$

$$\omega_{mstr'} = \pm\sqrt{\frac{f_H^2 k_y^2}{2k_h^2} - \frac{N^2}{2} + (B_0\cdot k)_h^2 - \sqrt{\frac{f_H^2 k_y^2}{4k_h^2}\left(\frac{f_H^2 k_y^2}{k_h^2} - 2N^2 + 4(B_0\cdot k)_h^2\right) + \frac{N^4}{4}}} \quad (8)$$

in the case of waves propagation in horizontal plane (k_x, k_y) dispersion relation (7) describes magnetostrophic waves

In the case of waves propagation along the vertical component of wave vector dispersion relation (7) describes vertical magnetic waves with frequency ω_{z_2}(4)

TRADITIONAL AND NON-TRADITIONAL β-PLANE APPROXIMATION

$$\begin{cases} \dfrac{\partial^2 u_x}{\partial y \partial t} - f_V \dfrac{\partial u_y}{\partial y} - \beta u_y + f_H \dfrac{\partial u_z}{\partial y} - \gamma u_z + \dfrac{\partial^2 P}{\partial y \partial x} + \dfrac{\partial}{\partial y}\left(B_y \dfrac{\partial B_y}{\partial x} + B_z \dfrac{\partial B_z}{\partial x} - B_y \dfrac{\partial B_x}{\partial y} - B_z \dfrac{\partial B_x}{\partial z}\right) = 0, \\ \dfrac{\partial u_y}{\partial t} + f_V u_x + \dfrac{\partial P}{\partial y} + B_x \dfrac{\partial B_x}{\partial y} + B_z \dfrac{\partial B_z}{\partial y} - B_x \dfrac{\partial B_y}{\partial x} - B_z \dfrac{\partial B_y}{\partial z} = 0, \\ \dfrac{\partial u_z}{\partial t} - f_H u_x + \dfrac{\partial P}{\partial z} + \rho' + B_x \dfrac{\partial B_x}{\partial x} + B_y \dfrac{\partial B_y}{\partial z} - B_x \dfrac{\partial B_z}{\partial x} - B_y \dfrac{\partial B_z}{\partial y} = 0, \\ \dfrac{\partial \vec{B}}{\partial t} - (\vec{B}_0 \nabla)\vec{u} = 0, \qquad \dfrac{\partial \rho'}{\partial t} + N^2 u_z = 0, \qquad \operatorname{div} \vec{u} = 0. \end{cases}$$

linearized equation for u_x

linearized equation for u_y

linearized equation for u_z

linearized equation for magnetic field, density and divergence-free condition for velocity

Background state:

$$\vec{u_0} = 0, \vec{B_0} = const,$$

$$\frac{\partial P_0}{\partial z} = -\bar{\rho}(z),\ \bar{\rho}(z) = N^2 z \frac{\widetilde{\rho_0}}{g}$$

N^2 – Brunt–Väisälä frequency

$$\rho' = \frac{\rho g}{\widetilde{\rho_0}},\ P = \frac{p}{\widetilde{\rho_0}},\ \vec{B} = \frac{\vec{b}}{\sqrt{4\pi\widetilde{\rho_0}}}$$

In β-plane approximation we suppose that Coriolis parameter $\vec{f} = 2\vec{\Omega}$ varies slightly with small latitude variations

traditional

$$\vec{f} = (0, 0, f_{V'}),$$
$$f_{V'} = 2\Omega \sin\theta_0 + 2\Omega \cos\theta_0\,(\theta - \theta_0)R = \\ = f_V + \beta y$$

non-traditional

$$\vec{f} = (0, f_{H'}, f_{V'}),$$
$$f_{V'} = f_V + \beta y,$$
$$f_{H'} = 2\Omega \cos\theta_0 - 2\Omega \sin\theta_0\,(\theta - \theta_0)R = \\ = f_H + \gamma y$$

! taking into account the horizontal component of the Coriolis force

SOLUTIONS OF DISPERSION EQUATION ON TRADITIONAL β-PLANE

$$k^2\omega^4 + \beta k_x\omega^3 - \omega^2\left[f_V^2k_z^2 - N^2k_h^2 + 2k^2(B_0\cdot k)^2\right] - \beta k_x\omega\left[(B_0\cdot k)^2 - N^2\right] + (B_0\cdot k)^2\left[k^2(B_0\cdot k)^2 - N^2k_h^2\right] = 0,\ \boldsymbol{k_h} = (k_x, k_y, 0) \quad \textbf{(9)}$$

$\boxed{k_z \ll \boldsymbol{k}}$ In the case of waves propagation in horizontal plane (k_x, k_y) dispersion equation (9) has solutions in form of magneto-Rossby waves (10), (11) and magnetic gravity wave

$$\omega_{mr_1} = -\frac{\beta k_x}{2k_h^2} + \frac{1}{2}\sqrt{\frac{\beta^2k_x^2}{k_h^4} + 4(B_0\cdot k)^2} \quad (10)$$

$$\omega_{mgr} = \pm\sqrt{(B_0\cdot k)_h^2 - N^2}$$

$$\omega_{mr_2} = -\frac{\beta k_x}{2k_h^2} - \frac{1}{2}\sqrt{\frac{\beta^2k_x^2}{k_h^4} + 4(B_0\cdot k)^2} \quad (11)$$

$\boxed{\boldsymbol{B_0} = \boldsymbol{0}}$ $\Rightarrow$ $\omega_R = -\dfrac{\beta k_x}{k_h^2}$

$\boxed{\boldsymbol{k_h} = k_y}$ $\Rightarrow$ $\omega_{a_y} = \pm B_{0y}k_y$ $\Leftarrow$ $\boxed{\boldsymbol{k_h} = k_y}$

has **no analogue** in neutral fluid dynamics $\Leftarrow$ $\boxed{\boldsymbol{B_0} = \boldsymbol{0}}$

$\boxed{\boldsymbol{k} = k_z}$ In the case of waves propagation along the vertical component of wave vector dispersion equation (9) describes vertical magnetic waves ω_{z_1}(2), ω_{z_2}(4)

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new

In low-frequency limit dispersion equation (9) has solution in form of three-dimensional magneto-Rossby wave (12)

$$\boldsymbol{\omega_{MR_{3D}}} \approx \frac{(\boldsymbol{B_0}\cdot\boldsymbol{k})^2\left[\boldsymbol{k}^2(\boldsymbol{B_0}\cdot\boldsymbol{k})^2 - \boldsymbol{N}^2\right]}{\boldsymbol{\beta k_x}\left[(\boldsymbol{B_0}\cdot\boldsymbol{k})^2 - \boldsymbol{N}^2\right]} \quad \textbf{(12)}$$

$\boxed{\boldsymbol{B_0} = \boldsymbol{0}}$

$$\omega_{R_{3D}} \approx \frac{N^2\beta k_x}{{f_v}^2k_z^2 - N^2k_h^2}$$

$\boxed{k_z \ll \boldsymbol{k}}$

$$\omega_R = -\frac{\beta k_x}{k_h^2}$$

$\boxed{k_z \ll \boldsymbol{k}}$

In the case of waves propagation in horizontal plane (k_x, k_y) dispersion relation (12) has a form of dispersion relation for magneto-Rossby wave in **shallow water** approximation (13)

$$\boldsymbol{\omega_{MR_{SW}}} \approx \frac{\boldsymbol{k_h^2}(\boldsymbol{B_0}\cdot\boldsymbol{k})_{\boldsymbol{h}}^2}{\boldsymbol{\beta k_x}} \quad \textbf{(13)}$$

SOLUTIONS OF DISPERSION EQUATION ON NON-TRADITIONAL β-PLANE

$$k^2\omega^4 + k_x\omega^3\left[\beta - \gamma\frac{k_z}{k_y}\right] - \omega^2\left[\left(f_H k_y + f_v k_z\right)^2 - N^2k_h^2 + 2k^2(B_0\cdot k)^2\right] - k_x\omega\left[(B_0\cdot k)^2\left(\beta - \gamma\frac{k_z}{k_y}\right) - \beta N^2\right] +$$
$$+(B_0\cdot k)^2\left[k^2(B_0\cdot k)^2 - N^2k_h^2\right] = 0, \qquad \boldsymbol{k_h} = (k_x, k_y, 0) \quad \textbf{(14)}$$

$\boldsymbol{k} = k_x$ In the case of waves propagation along the x-component of wave vector dispersion equation (14) has solutions in form of magneto-Rossby waves (10), (11)

$\boldsymbol{k} = k_y$

$$\omega_{mig_y} = \pm\sqrt{(1/2)\left(f_H^2/2 - N^2/2 + B_y^2k_y^2\right) + \sqrt{(f_H^2/2 - N^2/2)^2 + f_H^2B_y^2k_y^2}} \quad (15)$$

$$\omega_{mstr_y} = \pm\sqrt{(1/2)\left(f_H^2/2 - N^2/2 + B_y^2k_y^2\right) - \sqrt{(f_H^2/2 - N^2/2)^2 + f_H^2B_y^2k_y^2}} \quad (16)$$

In the case of waves propagation along the y-component of wave vector dispersion equation (14) has solutions in form of one dimensional magnetic inertio-gravity wave (15) and one-dimensional magnetostrophic wave (16)

$\boldsymbol{k} = k_z$ In the case of waves propagation along the vertical component of wave vector dispersion equation (14) describes vertical magnetic waves ω_{z_1}(2), ω_{z_2}(4)

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new

In low-frequency limit dispersion equation (14) has solution in form of three-dimensional magneto-Rossby wave (17)

$$\boldsymbol{\omega_{MR'_{3D}}} \approx \frac{(\boldsymbol{B_0\cdot k})^2\left[\boldsymbol{k^2(B_0\cdot k)^2 - N^2k_h^2}\right]}{\boldsymbol{k_x}\left[(\boldsymbol{B_0\cdot k})^2\left(\boldsymbol{\beta - \gamma\frac{k_z}{k_y}}\right) - \boldsymbol{\beta N^2}\right]} \quad \textbf{(17)}$$

$k_z \ll \boldsymbol{k}$

$\boldsymbol{B_0} = \boldsymbol{0}$

$$\omega_{R'_{3D}} \approx \frac{N^2\beta k_x}{\left(f_H k_y + f_v k_z\right)^2 - N^2k_h^2}$$

In the case of waves propagation in horizontal plane (k_x, k_y) dispersion relation (17) has a form of dispersion relation for magneto-Rossby wave in **shallow water** approximation (13)

$$\boldsymbol{\omega_{MR_{SW}}} \approx \frac{\boldsymbol{k_h^2(B_0\cdot k)_h^2}}{\boldsymbol{\beta k_x}} \quad \textbf{(13)}$$

$k_z \ll \boldsymbol{k}$

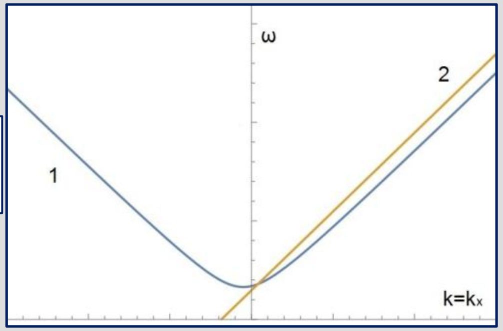
$$\omega_R = -\frac{\beta k_x}{k_h^2}$$

PHASE MATCHING CONDITION

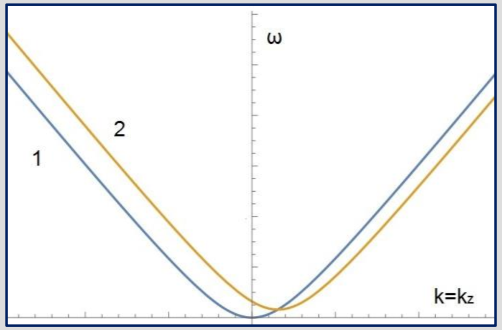
$$\omega_1(\boldsymbol{k}_1) + \omega_2(\boldsymbol{k}_2) = \omega_3(\boldsymbol{k}_3), \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3$$

The intersection of dispersion curves shifted from each other is an indicator of the satisfaction of the phase matching condition

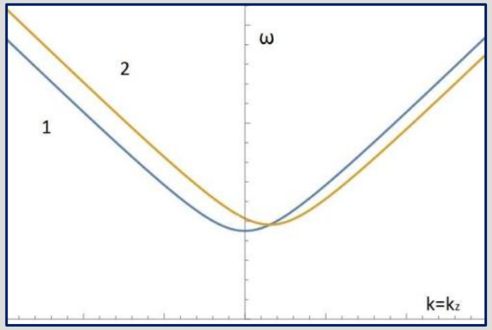
f-plane



$1: \omega_{mgr}, 2: \omega_a + \omega_{mgr}$

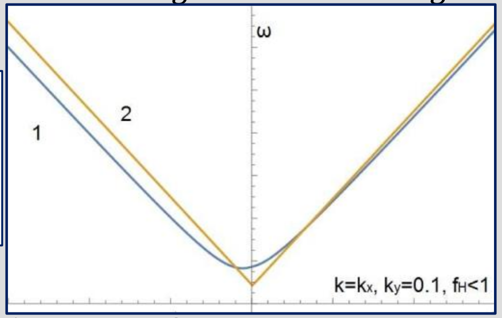


$1: \omega_{z_1}, 2: \omega_{z_2} + \omega_{z_1}$

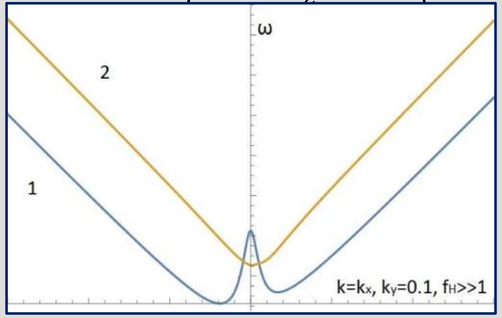


$1: \omega_{z_2}, 2: \omega_{z_2} + \omega_{z_2}$

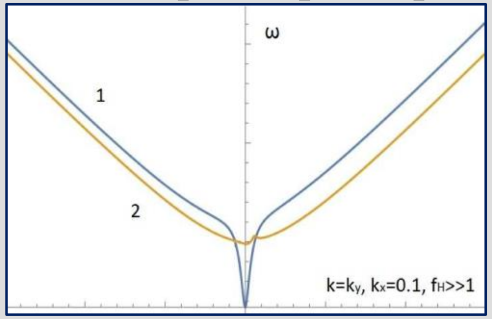
non-trad. f-plane



$1: \omega_{mig'}, 2: \omega_{mstr'} + \omega_{mstr'}$

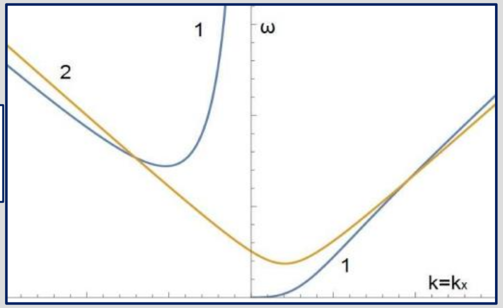


$1: \omega_{mig'}, 2: \omega_{mstr'} + \omega_{mig'}$

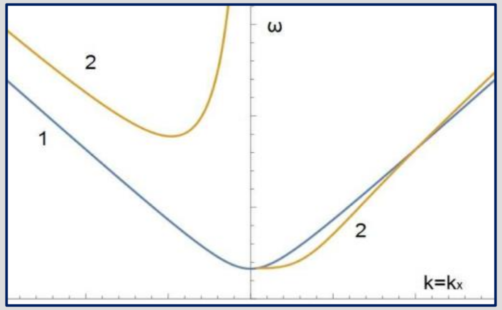


$1: \omega_{mig'}, 2: \omega_{mstr'} + \omega_{mig'}$

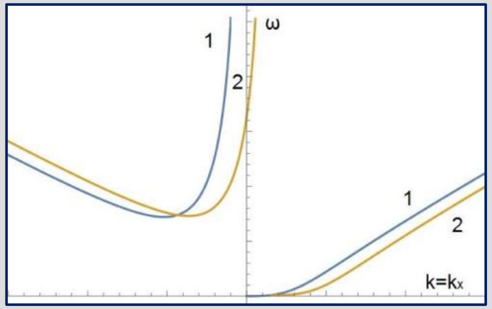
β-plane



$1: \omega_{mr_1}, 2: \omega_{mgr} + \omega_{mr_1}$

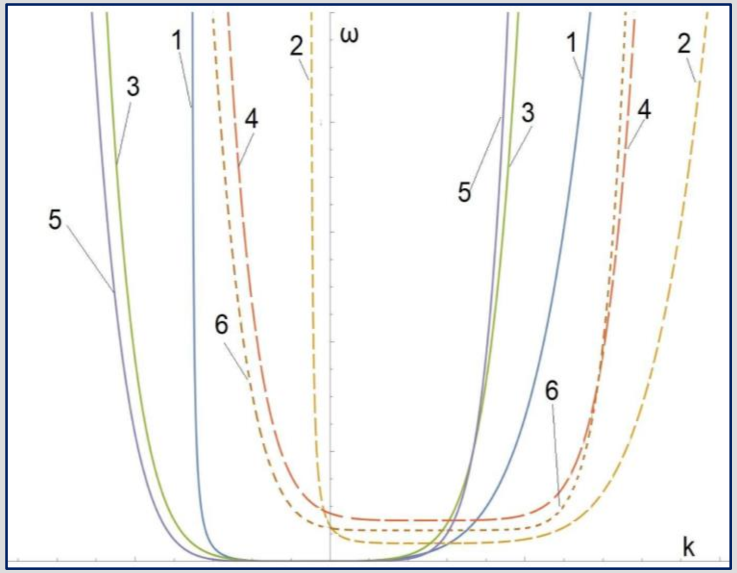


$1: \omega_{mgr}, 2: \omega_{mr_1} + \omega_{mgr}$



$1: \omega_{mr_1}, 2: \omega_{mr_1} + \omega_{mr_1}$

non-trad. β-plane



$1: \omega_{MR'_{3D}}, 2: \omega_{MR'_{3D}} + \omega_{MR'_{3D}}, k = k_x;$

$3: \omega_{MR'_{3D}}, 4: \omega_{MR'_{3D}} + \omega_{MR'_{3D}}, k = k_y;$

$5: \omega_{MR'_{3D}}, 6: \omega_{MR'_{3D}} + \omega_{MR'_{3D}}, k = k_z;$

MULTISCALE ASYMPTOTIC METHOD

Let us present the solution of the MHD Boussinesq equations in form of an **asymptotic series** in small parameter $\boldsymbol{\varepsilon}$

$$\vec{\boldsymbol{q}} = \overrightarrow{\boldsymbol{q_0}} + \boldsymbol{\varepsilon}\overrightarrow{\boldsymbol{q_1}} + \boldsymbol{\varepsilon}^2\overrightarrow{\boldsymbol{q_2}} + \cdots$$

$\overrightarrow{q_0}$ - stationary solution, $\overrightarrow{q_1}$ - linear solution, $\overrightarrow{q_2}$ - quadratic nonlinear effect

The equations that govern correction due to nonlinear effect can be obtained in the **second-order** approximation in parameter ε. The right-hand side of this equation contains **resonance terms** that lead to linear growth of the solution and violation of $\varepsilon^2 q_2 \ll \varepsilon q_1$ on a large scale

To avoid this, we introduce a **slowly-varying amplitude** that depends on slow time (T_1) and large space scales (X_1, Y_1, Z_1)

$$\vec{\boldsymbol{q}}(\boldsymbol{T_1}, \boldsymbol{X_1}, \boldsymbol{Y_1}, \boldsymbol{Z_1})\boldsymbol{e}^{\boldsymbol{i}(\boldsymbol{\omega T_0} - \boldsymbol{k_x X_0} - \boldsymbol{k_y Y_0} - \boldsymbol{k_z Z_0})}$$

(T_0, X_0, Y_0) - “fast’’ variables

(T_1, X_1, Y_1) - “slow’’ variables

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}; \frac{\partial}{\partial x} = \frac{\partial}{\partial X_0} + \varepsilon\frac{\partial}{\partial X_1}; \frac{\partial}{\partial y} = \frac{\partial}{\partial Y_0} + \varepsilon\frac{\partial}{\partial Y_1}; \frac{\partial}{\partial z} = \frac{\partial}{\partial Z_0} + \varepsilon\frac{\partial}{\partial Z_1}$$

COMPATIBILITY CONDITION

By means of our new form of solution we obtain system in the second-order approximation in parameter ε in following form

$$\boldsymbol{A(T_0,X_0,Y_0,Z_0)\overrightarrow{q_2} = -F(T_1,X_1,Y_1,Z_1)\overrightarrow{q_1} - G(T_0,X_0,Y_0,Z_0)(\overrightarrow{q_1},\overrightarrow{q_1})}$$

Now we can eliminate the resonant terms in the right-hand side using **compatibility condition:**

orthogonality of the right hand side **to the kernel of a linear operator on the left-hand side**

Multiplying the system by eigenvector of linear operator, we obtain the system in following form

$$\vec{z}A = 0 \quad \Rightarrow \quad -\vec{z}F\overrightarrow{q_1} - \vec{z}G(\overrightarrow{q_1},\overrightarrow{q_1}) = 0$$

$\vec{z}$ - eiegenvector for A

EQUATIONS FOR AMPLITUDES OF THREE INTERACTING WAVES

We introduce he solution in form of a **sum of three interacting waves**, satisfying phase matching condition

$$\boldsymbol{\overrightarrow{q_1} = \phi\vec{a}\left(\overrightarrow{k_1}\right)e^{i\vartheta_1} + \psi\vec{a}\left(\overrightarrow{k_2}\right)e^{i\vartheta_2} + \chi\vec{a}\left(\overrightarrow{k_3}\right)e^{i\vartheta_3} + c.c.}$$

Writing out successively terms proportional to $\boldsymbol{e^{i\vartheta_1}}$, $\boldsymbol{e^{i\vartheta_2}}$, and $\boldsymbol{e^{i\vartheta_3}}$, we obtain a system of equations that govern three amplitudes of interacting packets of waves in the Boussinesq approximation:

$$\begin{cases} s_1\phi = f_1\psi^*\chi \,, \\ s_2\psi = f_2\phi^*\chi \,, \\ s_3\chi = f_3\phi\psi \,. \end{cases}$$

$$s_i = r_i\frac{\partial}{\partial T_1} + p_i\frac{\partial}{\partial X_1} + q_i\frac{\partial}{\partial Y_1} + w_i\frac{\partial}{\partial Z_1}$$

differential operator with respect to the "slow" arguments

Coefficients f_i depend only on the initial conditions and characteristics of interacting waves.

Significant **differences** between the obtained amplitude equations for different Coloriolis approximations and different waves interactions are contained in **differential operators** s_i and **coefficients** f_i

PARAMETRIC INSTABILITIES

The system of amplitude equations for three interacting waves is universal system for describing parametric instabilities. Thus we have two types of parametric instabilities in magnetohydrodynamic flows of stratified rotating plasma in Boussinesq approximation for all Coriolis parameter approximations

First type realizes when one amplitude is large compared with another two at the initial moment

$$\boldsymbol{\phi} = \boldsymbol{\phi}_0 \gg \boldsymbol{\psi}, \boldsymbol{\chi}$$

decay of wave $\omega_1(k_1)$ into two waves $\omega_2(k_2)$ and $\omega_3(k_3)$ with instability increment

$$\boldsymbol{\Gamma} = \sqrt{\frac{|f_2 f_3|}{|r_2 r_3|}}\,|\boldsymbol{\phi}_0| > \mathbf{0}$$

Second type realizes when one amplitude is small compared with another two at the initial moment

$$\boldsymbol{\phi} \ll \boldsymbol{\psi} = \boldsymbol{\psi}_0, \boldsymbol{\chi} = \boldsymbol{\chi}_0$$

amplification of wave $\omega_1(k_1)$ by two waves $\omega_2(k_2)$ and $\omega_3(k_3)$ with instability increment

$$\boldsymbol{\Gamma} = \sqrt{\frac{|f_1|}{|r_1|}}\,|\boldsymbol{\psi}_0 \boldsymbol{\chi}_0| > \mathbf{0}$$

CONCLUSION

- Linearized magnetohydrodynamic equations in Boussinesq approximation for rotating stratified layer of plasma with linear density profile are solved in **four** cases of **Coriolis parameter approximation**
- Linear solutions are obtained in form of waves govern by **buoyancy, Coriolis and Lorentz** forces
  - Three-dimensional magnetic inertio-gravity and magnetostrophic waves
  - Low-frequency magneto-Rossby waves
- By means of **non-traditional** approximations influence of **horizontal component of Coriolis parameter** is studied in conjunction with effects of **stratification**
- Dispersion curves of all obtained wave types are studied to satisfy **phase matching condition**
- **Equations for amplitudes** of three interacting waves are obtained by means of multiscale asymptotic method for all identified types of interactions. **Parametric instabilities** are studied and their **increments** are found.

FOR MORE DETAILS

M. A. Fedotova, A. S. Petrosyan, Waves processes in three-dimensional stratified flows of rotating plasma in Boussinesq approximation, JETP, accepted in print (2020)