Ergodicity of a stochastic Two Layer Quasi Geostrophic Model

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Why ergodicity?

Given a stochastic model for the evolution of a physical quantity u

$$du(t) = F(u,t)dt + \sigma dW(t), \quad u(t_0) = u_0 \tag{1}$$

with W Wiener process and $F(u(\cdot),t)$ nonlinear, suppose it has a unique solution $u(t,\omega;u_0)$ depending continuously on u_0. Then it defines a **transition process** (or **semigroup**) as follows: given a continuous observable φ and an initial condition u_0

$$(P_t\varphi)(u_0) := \mathbb{E}[\varphi(u(t,\cdot;u_0))] \tag{2}$$

In practice the transition semigroup prescribes how observables evolve under the dynamics we are considering.

Why ergodicity?

Establishing ergodicity means knowing that the **time average** of an observable for large times is a very good approximation of its **spatial average**, or more precisely of its average with respect to a measure on the state space that is invariant with respect to the dynamic.

We call a measure μ **invariant** with respect the transition semigroup P_t if for all measurable subsets A

$$P_t^* \mu(A) = \mu(P_t \chi_A) = \mu(A)$$

where χ_A is the indicator function of A.

In practice to show ergodicity it is enough to prove **uniqueness** of such invariant measure.

Fluid dynamics models

For the Stochastic 2D Navier-Stokes (SNS) equation, under appropriate conditions, it has been established ([Da Prato and Zabczyk, 2014] and reference within):

- ✓ existence and uniqueness of solutions
- ✓ existence of an invariant measure
- ✓ uniqueness of the invariant measure, even with very degenerate noise ([Hairer and Mattingly, 2006]).

Do we have similar properties for **atmospheric and ocean models**?
Next we focus on a two layer quasi-geostrophic model driven by a stochastic forcing on the first layer to better represent the action of the **wind shear**.

Two layer quasi-geostrophic model + noise

Consider the quasi-geostrophic vorticity equations, with β-plane approximation, for the **streamfunctions** $\psi_1(x, y, t)$ and $\psi_2(x, y, t)$ on a squared domain with periodic boundary conditions

$$\begin{aligned} dq_1 + J(\psi_1, q_1 + \beta y)dt &= (\nu\Delta^2\psi_1 + f)dt + \sigma dW \\ \frac{\partial q_2}{\partial t} + J(\psi_2, q_2 + \beta y) &= \nu\Delta^2\psi_2 - r\Delta\psi_2 \end{aligned} \tag{3}$$

where q_1 and q_2 are the QG potential vorticities defined as

$$\begin{aligned} q_1 &= \Delta\psi_1 - F_1(\psi_1 - \psi_2) \\ q_2 &= \Delta\psi_2 - F_2(\psi_2 - \psi_1) \end{aligned}$$

- $J(a,b) = \nabla^{\perp} a \cdot \nabla b$;
- F_1, F_2 are positive constants depending on the thickness of the layers, the densities, the gravitational acceleration and the Coriolis parameter;
- $\nu > 0$ is the viscosity coefficient;
- $-r\Delta\psi_2$ represent the effect of bottom friction, $r > 0$;
- $f(x,y)$ is the mean deterministic wind forcing;
- σdW white noise describing the fluctuating part of the wind shear.

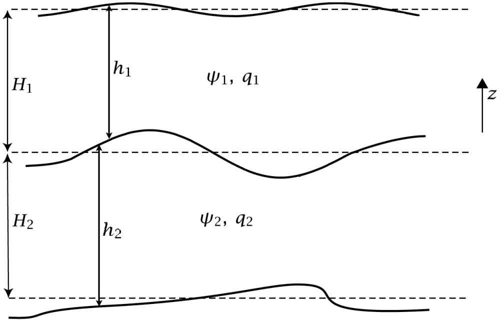

Figure: From [Vallis, 2006]

Disclaimer: the infinite dimensionality

How did we pass from dealing with an harmless SDE (1) to Stochastic PDE like Navier Stokes or (3)?

Recall that we can see a (stochastic) PDE as a (stochastic) ODE (1) defined on an appropriate space of functions, or, equivalently, as an infinite system of ODEs. For example, let $X \subset \mathbb{R}^2$ be the spatial domain, then, for a given realization of the noise ω, one can see the QG potential vorticities as

$$q_i(\omega) : [0, T] \mapsto L^2(X), \quad i = 1, 2.$$

$$\Downarrow$$

The state space of the associate dynamical system is **infinite dimensional.**

Model analysis

For the full model (3) we established:

- ✓ existence and uniqueness of solutions
- ✓ continuous dependence on the initial condition

Hence we can define the associated ∞-dimensional semigroup as in (2).

For such semigroup we proved:

- ✓ existence of an invariant measure (with a classic Krylov-Bogoliubov argument)
- ✓ uniqueness of the invariant measure (under a condition on parameters)

Hence, under appropriate conditions, the two layer quasi geostrophic model (3) is **ergodic**.

On the ergodicity proof: challenges

- Usual finite dimensional approaches do not generalize straight away;
- The infinite dimension makes hard to control the small scales;
- Stochastic forcing is not affecting all degrees of freedom as it acts directly **only** on the first layer. Can this be enough to drive the long time behaviour of second layer?

Currently we have a *conditional* positive answer to last question.

On the ergodicity proof: coupling approach

Idea 1: Solutions driven by the **same** noise realisation can be synchronised in finite time through an appropriate coupling (hard to achieve!)

Idea 2: Consider instead an **equivalent** noise = original noise + control term, and require synchronization only on the long run and with positive probability

The second approach is enough to establish the uniqueness of the invariant measure ([Glatt-Holtz et al., 2017]).

The main difficulty in using this technique for the two layer QG is to find such **control** that has to be finite dimensional and act only on the upper layer.

Exponential mixing

Mixing is a very desirable property for dynamical systems generating from physical applications as it brings a notion of **asymptotic independence** of the trajectories.

We may also say the system **forgets** (exponentially) **quickly** of its initial condition in this case. Namely that having two solutions with different initial conditions $\mathbf{q}(t;\mathbf{q}_0^{(1)})$ and $\mathbf{q}(t;\mathbf{q}_0^{(2)})$, their laws converge (exponentially) quickly to the invariant measure μ.

Under the same condition on the parameters and techniques similar to the asymptotic coupling and optimal control, we can also establish exponential mixing for the stochastic two layer QG model (3).

Outlook

Next steps:

- lift the condition on parameters;
- establish linear response theory as in [Hairer and Majda, 2010] to rigorously study whether changes to the mean long term behaviour of the system happen abruptly or gradually (i.e. differentiably) in response to changes in parameters of the dynamics.

Note that the result presented also holds (with simpler proofs) in cases where

- the diffusivity is stronger ($\nu \Delta^p \psi_i$, $p > 2$)
- the noise is in both layers

Questions? Comments? Suggestions?

References

- Da Prato, G. and Zabczyk, J. (2014). *Stochastic equations in infinite dimensions*. Cambridge university press.
- Glatt-Holtz, N., Mattingly, J. C., and Richards, G. (2017). On unique ergodicity in nonlinear stochastic partial differential equations. *Journal of Statistical Physics*, 166(3):618–649.
- Hairer, M. and Majda, A. J. (2010). A simple framework to justify linear response theory. *Nonlinearity*, 23(4):909.
- Hairer, M. and Mattingly, J. C. (2006). Ergodicity of the 2d navier-stokes equations with degenerate stochastic forcing. *Annals of Mathematics*, 164(3):993–1032.
- Vallis, G. K. (2006). *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press, Cambridge, U.K.