Ridge estimation iterative algorithm to ill-posed uncertainty adjustment model

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Contents

1. Introduction

- **Uncertainty is a general term for concepts such as inaccuracy, fuzziness, and ambiguity.**
- **The uncertainty can be expressed by variance, mean square error, error interval and error ellipsoid.**
- **The research of uncertainty evaluation methods, and the research to effectively reduce the impact of uncertainty have become hot topics**
- **As for the ill-posed problem of uncertainty adjustment model, there are few related researches on how to solve the problem.**

- **This paper establishes the corresponding ill-conditioned uncertainty adjustment model and the adjustment criterion.**
- **The iterative solution formula of the ill-conditioned uncertainty estimation method is derived and the algorithm is validated and discussed through examples.**

2.Uncertainty adjustment model and adjustment criterion

- **Uncertain adjustment model is**

$$\boldsymbol{L}+\Delta\boldsymbol{L}=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{X}+\boldsymbol{e}$$

$$\left\|\boldsymbol{e}_{\boldsymbol{A}}\right\|_2\le\varphi$$

$$\left\|\Delta\boldsymbol{L}\right\|_2\le\beta$$

- **The uncertainty least squares (ULS) adjustment criterion is established as**

$$\min = f(\hat{\boldsymbol{e}},\Delta\hat{\boldsymbol{L}},\hat{\boldsymbol{e}}_{\boldsymbol{A}})=\left\|\hat{\boldsymbol{e}}\right\|_2^2+\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2+\left\|\hat{\boldsymbol{e}}_{\boldsymbol{A}}\right\|_2^2+\mu(\left\|\hat{\boldsymbol{e}}_{\boldsymbol{A}}\right\|_2^2-\varphi^2)+u(\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2-\beta^2)$$

- **Constructing a generalized Lagrangian objective function**

$$F(\hat{\boldsymbol{e}},\Delta\hat{\boldsymbol{L}},\hat{\boldsymbol{e}}_{\boldsymbol{A}},\lambda)=\left\|\hat{\boldsymbol{e}}\right\|_2^2+\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2+\left\|\hat{\boldsymbol{e}}_{\boldsymbol{A}}\right\|_2^2+2\boldsymbol{\lambda}^{\mathrm{T}}[\boldsymbol{L}+\Delta\hat{\boldsymbol{L}}-\hat{\boldsymbol{e}}-(\boldsymbol{A}+\Delta\hat{\boldsymbol{A}})\hat{\boldsymbol{X}}]+\mu(\left\|\hat{\boldsymbol{e}}_{\boldsymbol{A}}\right\|_2^2-\varphi^2)+u(\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2-\beta^2)$$

- The following normal equation is obtained

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{L}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\hat{\boldsymbol{X}}=\boldsymbol{A}^{\mathrm{T}}\lambda(\frac{1}{1+\mu}\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+\frac{1}{1+u}+1)$$

- Let

$$\hat{\boldsymbol{\eta}}^{(i)}=\frac{(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})^{\mathrm{T}}(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})}{(1+\mu^{(i)})(\frac{1}{1+\mu^{(i)}}\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\hat{\boldsymbol{X}}^{(i)}+\frac{1}{1+u^{(i)}}+1)}$$

- we can transform equation into an iterative form

$$\hat{\boldsymbol{X}}^{(i+1)}=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{L}+\hat{\boldsymbol{X}}^{(i)}\hat{\boldsymbol{\eta}}^{(i)})$$

- The iteration is terminated when

$$\left\|\hat{\boldsymbol{X}}^{(i+1)}-\hat{\boldsymbol{X}}^{(i)}\right\|<\varepsilon$$

3.Ridge estimation of ill-conditioned uncertainty adjustment model

- In order to reduce the effect of ill-posed problem on the adjustment uncertainty model, a stability function is added into ULS adjustment criterion

$$f(\hat{\boldsymbol{e}},\Delta\hat{\boldsymbol{L}},\hat{\boldsymbol{e}}_{\boldsymbol{A}})=\|\hat{\boldsymbol{e}}\|_2^2+\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2+\|\hat{\boldsymbol{e}}_{\boldsymbol{A}}\|_2^2+\alpha\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}=\min$$

- Where α is the ridge parameter.

- Lagrangian target function is constructed as

$$F(\hat{\boldsymbol{e}},\Delta\hat{\boldsymbol{L}},\hat{\boldsymbol{e}}_{A},\lambda)=\|\hat{\boldsymbol{e}}\|_2^2+\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2+\|\hat{\boldsymbol{e}}_{A}\|_2^2+\alpha\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+2\boldsymbol{\lambda}^{\mathrm{T}}[\boldsymbol{L}+\Delta\hat{\boldsymbol{L}}-\hat{\boldsymbol{e}}-(\boldsymbol{A}+\Delta\hat{\boldsymbol{A}})\hat{\boldsymbol{X}}]+\mu(\|\hat{\boldsymbol{e}}_{A}\|_2^2-\varphi^2)+u(\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2-\beta^2)$$

- The first-order partial derivatives are deduced as

$$\frac{\partial F}{\partial \hat{\boldsymbol{e}}} = 2\hat{\boldsymbol{e}}^{\mathrm{T}} - 2\boldsymbol{\lambda}^{\mathrm{T}} = 0$$

$$\frac{\partial F}{\partial \Delta\hat{\boldsymbol{L}}} = 2\Delta\hat{\boldsymbol{L}}^{\mathrm{T}} + 2\boldsymbol{\lambda}^{\mathrm{T}} + 2u\Delta\hat{\boldsymbol{L}}^{\mathrm{T}} = 0$$

$$\frac{\partial F}{\partial \hat{\boldsymbol{e}}_{\boldsymbol{A}}} = 2\hat{\boldsymbol{e}}_{\boldsymbol{A}}{}^{\mathrm{T}} - 2\lambda^{\mathrm{T}}(\hat{\boldsymbol{X}}^{\mathrm{T}} \otimes \boldsymbol{I}_n) + 2\mu\hat{\boldsymbol{e}}_{\boldsymbol{A}}{}^{\mathrm{T}} = 0$$

$$\frac{\partial F}{\partial \hat{\boldsymbol{X}}} = 2\alpha\hat{\boldsymbol{X}}^{\mathrm{T}} - 2\lambda^{\mathrm{T}}(\boldsymbol{A} + \Delta\hat{\boldsymbol{A}}) = 0$$

$$\mu\left(\left\|\hat{\boldsymbol{e}}_{\boldsymbol{A}}\right\|_2^2 - \varphi^2\right) = 0$$

$$u\left(\left\|\Delta\hat{\boldsymbol{L}}\right\|_2^2 - \beta^2\right) = 0$$

- The normal equation is

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{L}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\hat{\boldsymbol{X}}=\boldsymbol{A}^{\mathrm{T}}\lambda\left(\frac{1}{1+\mu}\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+\frac{1}{1+u}+1\right)$$

- Also as

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{L}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\hat{\boldsymbol{X}}=\alpha\hat{\boldsymbol{X}}\cdot\left(\frac{1}{1+\mu}\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+\frac{1}{1+u}+1\right)-\frac{\hat{\boldsymbol{X}}(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}})^{\mathrm{T}}(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}})}{(1+\mu)\left(\frac{1}{1+\mu}\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+\frac{1}{1+u}+1\right)}$$

- When φ gets close to positive infinity and β is equal to zero, equation becomes the total least squares solution. that is

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{L}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\hat{\boldsymbol{X}}=\alpha(\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+1)\cdot\hat{\boldsymbol{X}}-\frac{\hat{\boldsymbol{X}}(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}})^{\mathrm{T}}(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}})}{(\hat{\boldsymbol{X}}^{\mathrm{T}}\hat{\boldsymbol{X}}+1)}$$

- Before the calculation, it is necessary to determine the ridge parameter α , which is usually determined by the L-curve method.

4.The procedure for the ridge estimation of the ill-conditioned uncertainty adjustment model

- the calculation steps are summarized as follows:
- ①The data preparation includes the coefficient matrix A, the observation vector L, and the uncertainties φ and β .
- ②The initial value $\hat{X}^{(0)}$ is set by using the least squares estimate or other estimate.
- ③The value of α is selected in the range of [a, b] by setting a step size of Δd .
- ④ Determine u and μ by equation

$$u=\frac{\left\|\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)}\right\|_2}{\beta}-\frac{\varphi\left\|\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)}\right\|_2\hat{\boldsymbol{X}}^{(k)\mathrm{T}}\hat{\boldsymbol{X}}^{(i)}}{\beta\left\|(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\right\|_2}-2 \qquad \mu=\frac{\left\|(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\right\|_2}{\varphi}-\frac{\beta\left\|(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\right\|_2}{\varphi\left\|\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)}\right\|_2}-\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\hat{\boldsymbol{X}}^{(i)}-1$$

- ⑤The parameter estimation is calculated as

$$\hat{\boldsymbol{X}}^{(i+1)}=[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+\alpha(\frac{1}{1+\mu^{(i)}}\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\hat{\boldsymbol{X}}^{(i)}+\frac{1}{1+u^{(i)}}+1)\boldsymbol{I}_n]^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{L}+\hat{\boldsymbol{X}}^{(i)}\hat{\boldsymbol{v}}^{(i)})$$

- Where

$$\hat{\boldsymbol{v}}^{(i)}=\frac{(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})^{\mathrm{T}}(\boldsymbol{L}-\boldsymbol{A}\hat{\boldsymbol{X}}^{(i)})}{(1+\mu^{(i)})(\frac{1}{1+\mu^{(i)}}\hat{\boldsymbol{X}}^{(i)\mathrm{T}}\hat{\boldsymbol{X}}^{(i)}+\frac{1}{1+u^{(i)}}+1)}$$

- ⑥.Draw the L-curve of

$$\frac{\left\|\boldsymbol{A}\hat{\boldsymbol{X}}^{(i+1)}-\boldsymbol{L}\right\|_2^2}{\frac{1}{1+\mu^{(i)}}\hat{\boldsymbol{X}}^{(i+1)\mathrm{T}}\hat{\boldsymbol{X}}^{(i+1)}+\frac{1}{1+u^{(i)}}+1}$$

- And

$$\hat{\boldsymbol{X}}^{(i+1)\mathrm{T}}\hat{\boldsymbol{X}}^{(i+1)}$$

- The obtained parameter estimation is considered as the ill-conditioned uncertainty ridge estimation solution.

5.Examples and analysis

- **Example 1**
- The condition number of $A^T A$ is is 2.0838×10^4, which means the ill-posed problem is severe.
- There are 5 unknown parameters

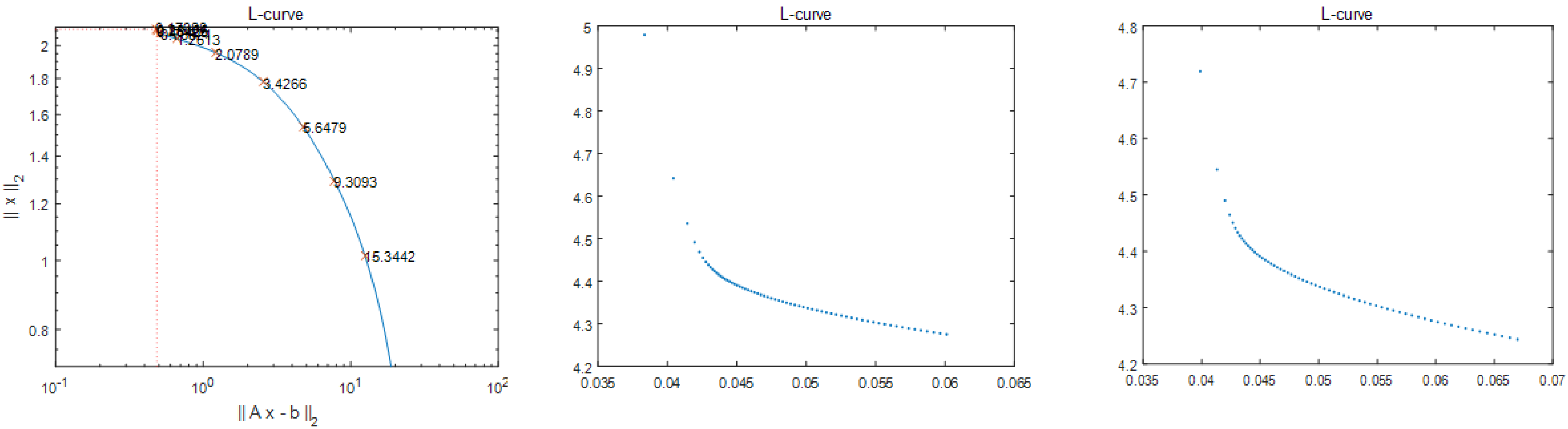
$$\boldsymbol{X} = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^{\mathrm{T}}$$

- their true values are $\boldsymbol{X} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]^{\mathrm{T}}$
- In order to verify the ill-conditioned least squares ridge estimation method, the LS, TLS, ULS, ridge estimation LS (R-LS), ridge estimation TLS (R- TLS) and ridge estimation ULS (R- ULS) are used to calculate the parameters.

- Results of different estimations

Program	Ture value	LS	TLS	ULS	R-LS	R-TLS	R-ULS
	1	1.3944	3.3051	2.8088	1.2157	1.2102	1.2032
$\hat{X}$	1	0.1223	-2.8048	-2.0432	0.3728	0.3776	0.3828
	1	0.7791	0.0596	0.2471	0.8280	0.8261	0.8225
	1	0.2628	-3.5894	-2.5883	0.5983	0.6034	0.6079
	1	1.4414	2.9034	2.5230	1.3157	1.3131	1.3103
$\left\|\Delta\hat{X}\right\|$	0	1.3088	6.7350	**5.3194**	0.8547	0.8468	**0.8389**

- The ridge parameter is determined by the L-curve method. The change of ridge parameter is shown in Fig.



（a） LS （b） TLS （c） ULS

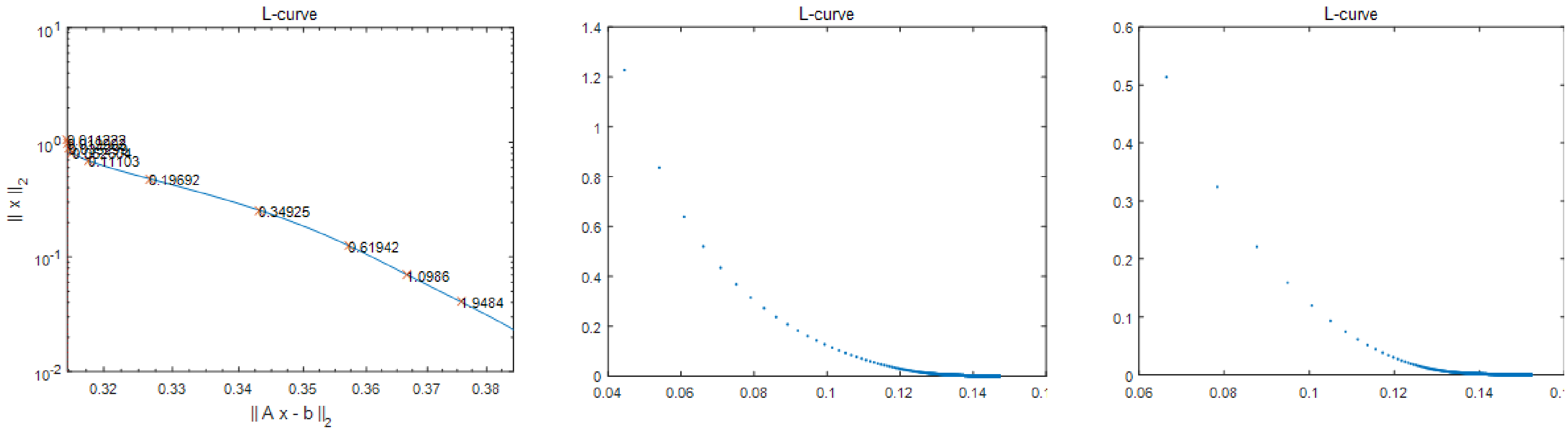
Fig1 The L-curve of each methods

- **Example 2**
- A example involving the spatial control network is conducted. P1, P2, ... and P10 are the 10 known points.P11, P12 and P13 are 3 unknown points.
- The simulated true values of the coordinates of the 3points are (0,0,0), (68,-26,9) and (14,41,-11)
- In order to verify the ill-conditioned least squares ridge estimation method, LS, TLS, ULS, R-LS, R-TLS and R-ULS are used to solve the problem.

- The effects of different uncertainties on the results are compared. The results are shown in Table.

Program	Ture value	LS	TLS	ULS	R-LS	R-TLS	R-ULS
	0	0.0530			0.0524	0.0444	0.0412
	0	-0.0846	divergent	divergent	-0.0698	-0.0228	-0.0126
	0	-0.8053			-0.6819	-0.1637	-0.0575
$\hat{x}$	68	68.0400	/	/	68.0406	68.0395	68.0313
	-26	-26.0303			-25.9053	-25.9129	-25.9305
	9	8.5113			8.9472	8.9616	8.9644
	14	14.0072			14.0078	14.0067	14.0084
	41	40.8080			40.9453	40.9654	40.9681
	-11	-11.5857			-11.1316	-11.0284	-11.0122
$\left\Vert \Delta\hat{x} \right\Vert$	0	1.1313			0.7116	0.2048	**0.1161**

- The ridge parameter is determined by the L-curve method. The change of ridge parameter is shown in Fig.



（a）LS （b）TLS （c）ULS

Fig2 The L-curve of each methods

Analysis of examples

- Both the TLS and the ULS are more severely affected by ill-posed problem compared to the LS. In other word, they are more sensitive to ill-posed problem.
- The optimal difference norms in the two examples are 0.8389 and 0.1161, which are smaller than those of the R-LS and R-TLS. This indicates that the ridge estimation is useful for the ill-conditioned uncertainty adjustment model.
- The results indicate that the solution of the ridge estimation for the ill-conditioned uncertainty adjustment model is constrained by the bounded uncertainty.
- It demonstrates that with the increase of uncertainty, the sensitivity of the ridge estimation for the ill-conditioned uncertainty adjustment model to the uncertainty is decreased.

6.Conclusion

- The ridge estimation is introduced, and the uncertainties of both the coefficient matrix and the observation vector are taken into consideration, deriving the adjustment criterion of the ridge estimation for the ill-conditioned uncertainty adjustment model.
- The corresponding iterative algorithm is formulated.
- The experimental results show that the R-ULS can efficiently limit the influence of ill-posed problem and reduce the difference norm.

Thanks for your attention!