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Hidden Magnetizations and Localization Constraints

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Problem Description

The (lithospheric) magnetic field produced by a vertically integrated magnetization (VIM) $\mathbf{m}$ is given by

$$\mathbf{B}_{lith}(x) = \nabla \int_{\mathbb{S}_R} \mathbf{m}(y) \cdot \frac{x-y}{|x-y|^3} d\omega(y)$$

The **corresponding inverse problem** is: Knowing $\mathbf{B}_{lith}$, what can we say about $\mathbf{m}$?

Any such magnetization can be decomposed by the Hardy-Hodge decomposition

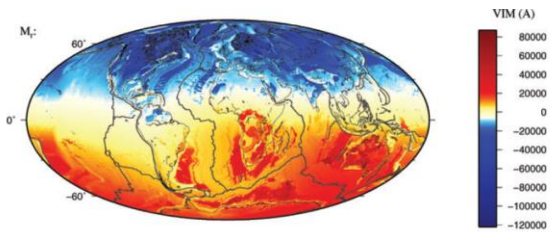
$$\mathbf{m} = \mathbf{m}_+ + \mathbf{m}_- + \mathbf{m}_{df}.$$

Only the $\mathbf{m}_+$-contribution can be reconstructed uniquely, i.e., it is **'visible'**. The **'invisible/hidden'** magnetizations are characterized by the nullspace of the operator $B : L^2(\mathbb{S}_R, \mathbb{R}^3) \to L^2(\mathbb{S}_R^{ext})$, $\mathbf{m} \mapsto \int_{\mathbb{S}_R} \mathbf{m}(y) \cdot \frac{\cdot - y}{|\cdot - y|^3} d\omega(y)$. In other words, the hidden magnetizations (i.e., the $\mathbf{m}_-$ and $\mathbf{m}_{df}$-contributions) do not produce any magnetic field in the exterior of the sphere $\mathbb{S}_R$ and can, therefore, not be reconstructed from knowledge of $\mathbf{B}_{lith}$.

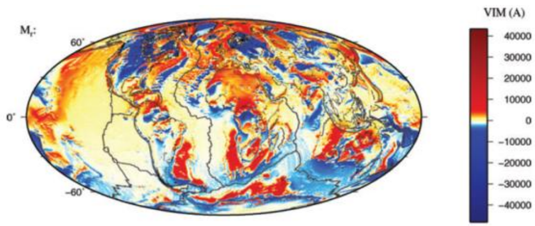
The **problem that we are addressing** is: What assumptions on $\mathbf{m}$ can be made in order to be able to reconstruct more than just the $\mathbf{m}_+$-contribution?

Problem Description

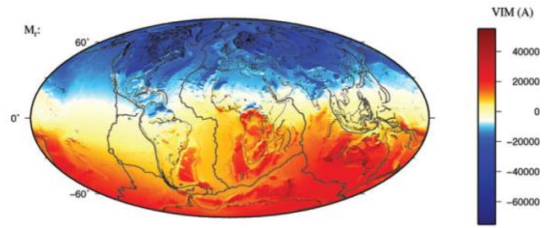
Total magnetization



'visible' magnetization

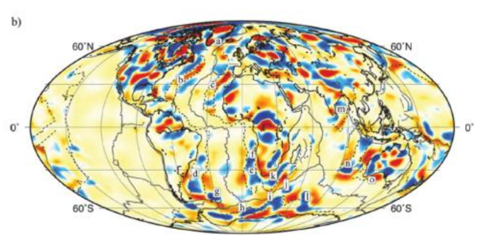


'invisible/hidden' magnetization



Magnetization model and corresponding magnetic field from **Gubbins et al.: Analysis of lithospheric magnetisation in vector spherical harmonics, GJI 187 (2011)**. In this example, no inverse problem is solved; the magnetization model has been compiled from geological and tectonical information. It serves merely as an illustration on which information on $\mathbf{m}$ cannot be recovered in the general inverse problem.

Forward Model of Magnetic Field



Assumption 1: Magnetization is Spatially Localized

If is known that in advance that the **magnetization is spatially localized in some suberegion** Γ of the sphere, i.e., $\mathbf{m}(x) = 0$ for $x \in \mathbb{S}_R \setminus \Gamma$, then the **$\mathbf{m}_+$- and $\mathbf{m}_-$-** contribution are determined uniquely by knowledge of $\mathbf{B}_{lith}$. It is not necessary to know Γ precisely.

An approximation of $\mathbf{m}$ can be obtained by computing a minimizer of

$$\min_{\mathbf{m}\in H_2(\mathbb{S}_R,\mathbb{R}^3)} \|\mathbf{B}_{lith} - \nabla B[\mathbf{m}]\|^2_{L^2(\mathbb{S}_R^{ext},\mathbb{R}^3)} + \lambda_1 \|\mathbf{m}\|^2_{H_2(\mathbb{S}_R,\mathbb{R}^3)} + \lambda_2 \|\mathbf{m}\|^2_{L^2(\mathbb{S}_R\setminus\Gamma,\mathbb{R}^3)}.$$

The parameter $\lambda_1 > 0$ enforces a typical H_2-regularization of the otherwise exponentially ill-posed problem while $\lambda_2 > 0$ enforces localization within Γ. More details on this approach can be found in **Gerhards: On the Unique Reconstruction of Induced Spherical Magnetizations, Inverse Problems 32 (2016)**.

Assumption 1: Magnetization is Spatially Localized

An **alternative approach** to incorporating localization constraints via solving the minimization problem on the previous slide is given by computing the $\mathbf{m}_+$-contribution by inversion of $\mathbf{B}_{lith}$ without constraints (as it is done frequently in the geomagnetic community) and then use localization constraints to obtain $\mathbf{m}_-$ from $\mathbf{m}_+$.

The latter can be achieved by solving a system of linear equations of the form

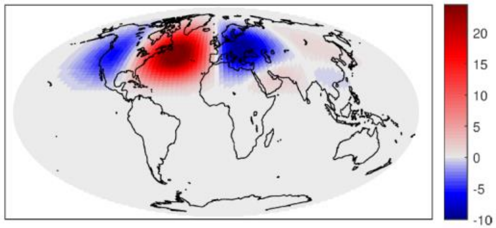
$$\left(\mathbf{L}_2^T\mathbf{L}_2 + \lambda\mathbf{R}^T\mathbf{R}\right)\begin{pmatrix}\hat{\mathbf{m}}_- \\ \hat{\mathbf{m}}_{df}\end{pmatrix} = \mathbf{L}_2^T\mathbf{L}_1\hat{\mathbf{m}}_+.$$

Here, $\hat{\mathbf{m}}_-$, $\hat{\mathbf{m}}_+$, and $\hat{\mathbf{m}}_{df}$ denote vectors of spherical Fourier coefficitents of the corresponding contributions of $\mathbf{m}$. And $\mathbf{L}_2$ denotes the matrix whose columns contain the spherical Fourier coeffcients of the restrictions of $\mathbf{y}_{n,k}^-$ and $\mathbf{y}_{n,k}^{df}$ to $\mathbb{S}_R \setminus \Gamma$, while $\mathbf{L}_1$ contains the spherical Fourier coeffcients of the restrictions of $\mathbf{y}_{n,k}^+$ to $\mathbb{S}_R \setminus \Gamma$; with $\mathbf{y}_{n,k}^{+/-/df}$ denoting corresponding vector spherical harmonics of degree n and order k. The matrix $\mathbf{R}$ reflects a typical H_2-regularization term.

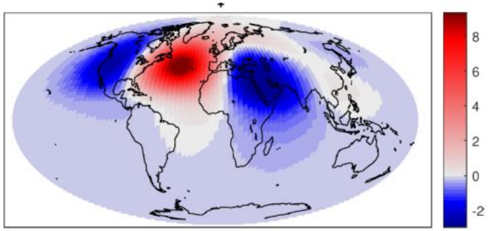
More details on this approach can be found in **Gerhards: A Brief Note on Computing Silent from Non-Silent Contributions of Spatially Localized Magnetizations on a Sphere, Preprint (2020)**.

Assumption 1: Magnetization is Spatially Localized

total magnetization

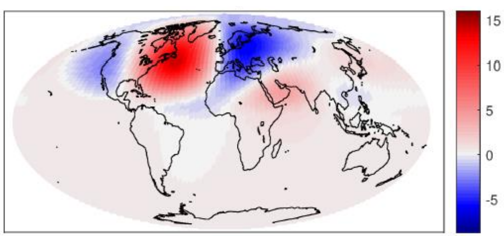


known 'visible' magnetization

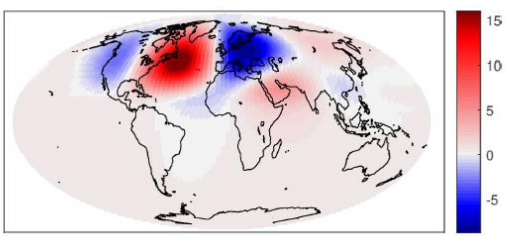


The top image shows a toy example of the visible contribution $\mathbf{m}_+$ that we assume to know, for an (unknown) total magnetization $\mathbf{m}$ that is localized in the upper hemisphere. The bottom images show reconstructions of the $\mathbf{m}_-$-contribution of the total magnetization via the alternative approach from the previous slide, with varying regularization parameters λ and the (unknown) true $\mathbf{m}_-$ at the very bottom.

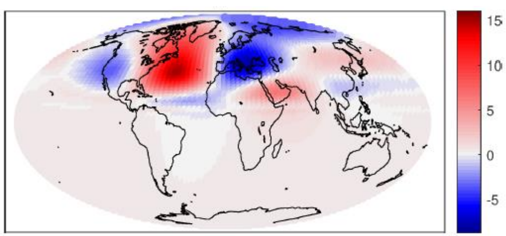
$\lambda=10^{-12}$



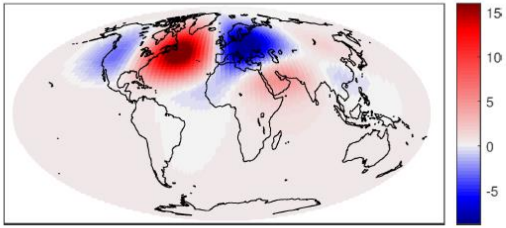
$\lambda=10^{-15}$



$\lambda=10^{-17}$



(unknown) 'invisible/hidden' magnetization



Assumption 2: Magnetization is Induced

Let us assume that the magnetization is of induced form and remanent contributions are neglected, i.e.,

$$\mathbf{m} = \chi \mathbf{B}_{ind}.$$

Here, χ denotes a vertically integrated susceptibility (VIS) and $\mathbf{B}_{ind}$ an ambient inducing magnetic field (in the case of the Earth, e.g., the core field). In **Maus and Haak: Magnetic field annihilators – invisible magnetization at the magnetic equator, GJI 155 (2003)** it has been shown that, even if the inducing field $\mathbf{B}_{ind}$ is known in advance, the susceptibility **cannot** be reconstructed uniquely from knowledge of $\mathbf{B}_{lith}$.

In **Lesur and Vervelidou: Retrieving lithospheric magnetization distribution from magnetic field models, GJI 220 (2020)** it has been shown that, if $\mathbf{B}_{ind}$ is assumed to be unknown but constant, then there exist methods to derive candidates for the direction of $\mathbf{B}_{ind}$. However, an inducing magnetic field of constant direction can be reasonable for regional studies but it is not realistic at global scales.

Assumption 3: Magnetization is Induced and Localized

We again assume that the magnetization is of induced form and remanent contributions are neglected, i.e.,

$$\mathbf{m} = \chi \mathbf{B}_{ind}.$$

Additionally, we assume that the susceptibility χ is spatially localized in a subregion Γ of $\mathbb{S}_R$, i.e., $\chi(x) = 0$ for $x \in \mathbb{S}_R \setminus \Gamma$. In that case, it has been shown in **Gerhards: On the Unique Reconstruction of Induced Spherical Magnetizations, Inverse Problems 32 (2016)** that χ is determined uniquely by knowledge of $\mathbf{B}_{lith}$. For the Earth, e.g., the inducing field $\mathbf{B}_{ind}$ might be given by the core magnetic field, but the assumption that the susceptibility is spatially localized is not realistic. In other setups, the assumption of spatial localization might be more realistic, but the inducing magnetic field will typically not be known.

Therefore, let us assume that the inducing field is a dipole field, i.e.,

$$\mathbf{B}_{ind}(x) = \frac{3(x \cdot \mathbf{d})x - \mathbf{d}|x|^2}{|x|^5},$$

with unknown dipole direction $\mathbf{d}$. In **Gerhards: On the reconstruction of inducing dipole directions and susceptibilities from knowledge of the magnetic field on a sphere, Inv. Prob. Sci. Engin. 27 (2019)** it has been shown that the problem is unique for most geophysical situations and that candidates for the direction $\mathbf{d}$ can be computed and subsequently $\mathbf{B}_{lith}$ can be inverted for χ.

Assumption 3: Magnetization is Induced and Localized

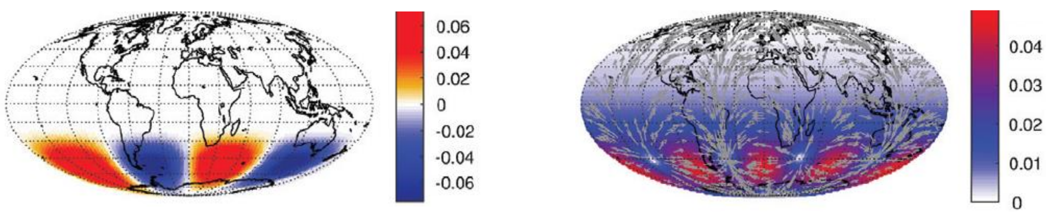
The procedure of finding $\mathbf{d}$ is as follows: First, reconstruct the contributions $\mathbf{m}_+$ and $\mathbf{m}_-$ by solving

$$\min_{\mathbf{m}\in H_2(\mathbb{S}_R,\mathbb{R}^3)} \|\mathbf{B}_{lith} - \nabla B[\mathbf{m}]\|^2_{L^2(\mathbb{S}_R^{ext},\mathbb{R}^3)} + \lambda_1\|\mathbf{m}\|^2_{H_2(\mathbb{S}_R,\mathbb{R}^3)} + \lambda_2\|\mathbf{m}\|^2_{L^2(\mathbb{S}_R\setminus\Gamma,\mathbb{R}^3)}. \quad (1)$$

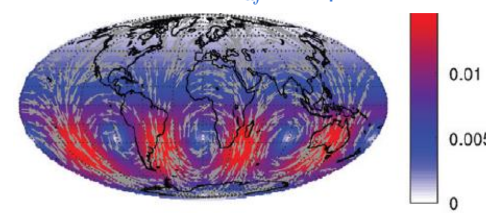
The contribution $\mathbf{m}_{df}$ is not determined uniquely by this but it is also not required for the further procedure. The unqiue contributions $\mathbf{m}_+$ and $\mathbf{m}_-$ are subsequently used to compute a **particular polynomial $P_\mathbf{m}$ whose zeros deliver candidates for the dipole direction $\mathbf{d}$**. With this $\mathbf{d}$ we obtain the inducing magnetic field and can determine the susceptibility by solving

$$\min_{\chi\in H_2(\mathbb{S}_R)} \|\mathbf{B}_{lith} - \nabla B[\chi\mathbf{B}_{ind}]\|^2_{L^2(\mathbb{S}_R^{ext},\mathbb{R}^3)} + \lambda_1\|\chi\|^2_{H_2(\mathbb{S}_R)} + \lambda_2\|\chi\|^2_{L^2(\mathbb{S}_R\setminus\Gamma)}. \quad (2)$$

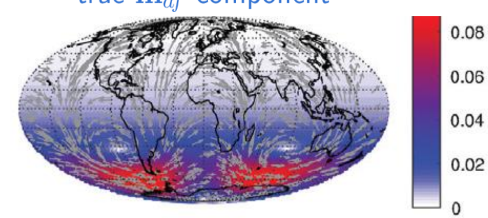
reconstructed $\mathbf{m}_+$ and $\mathbf{m}_-$ components



reconstructed $\mathbf{m}_{df}$ component



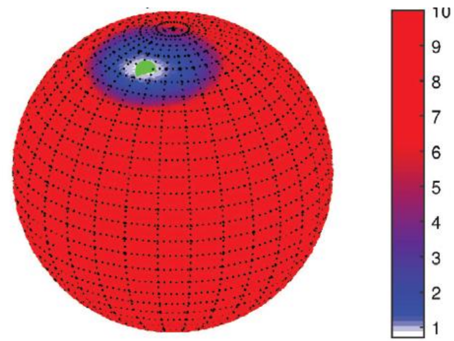
true $\mathbf{m}_{df}$ component



Toy example: Reconstructed contributions of the magnetization via (1). The true $\mathbf{m}_{df}$-component is shown at the bottom right. It is clearly different from the reconstructed component, but this contribution is not required for the determination of $\mathbf{d}$.

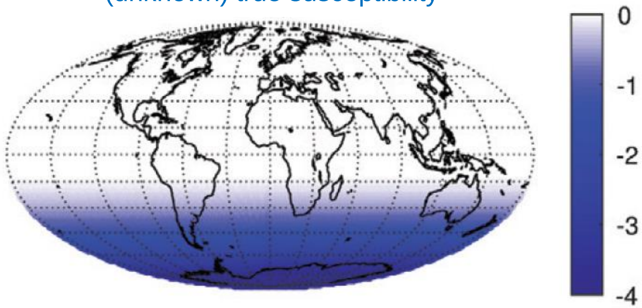
Assumption 3: Magnetization is Induced and Localized

Polynomial $P_{\mathbf{m}}$ based on the previous reconstruction of $\mathbf{m}$, and the obtained dipole direction $\mathbf{d}$ (indicated by the green dot).

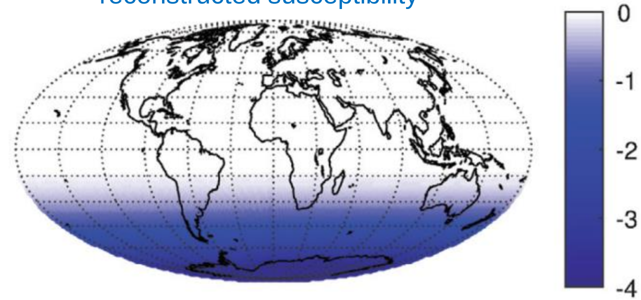


Susceptibility χ reconstructed from (2) and knowledge of the previously obtained $\mathbf{d}$.

(unknown) true susceptibility



reconstructed susceptibility



Conclusion

The inverse problem of recovering vertically integrated magnetizations and susceptibilities from (geo)magnetic field data has been more throrougly studied in the last few years with respect to its uniqueness, and how to overcome some of the non-uniqueness.

Some spectral methods have recently been tested succesfully with realistic magnetization models.

The presented approaches based on spatial localization have, so far, only been applied to toy examples and will be tested with realistic models in the near future.

The advantage of spatial localization over purely spetral methods is that an assumption such as the magnetization $\mathbf{m}$ being of induced type is not necessary to obtain more thn just the $\mathbf{m}_+$-contribution.