HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

Effect of finite correlation time on the wave-particle interactions of nonlinear electrostatic structures with electrons in the Earth's radiation belts.

Adnane Osmane w/ Aleksi Piispa & Atte Martikainen
Department of Physics, University of Helsinki, Finland

EGU General Assembly 2020

@Adnane_Osmane

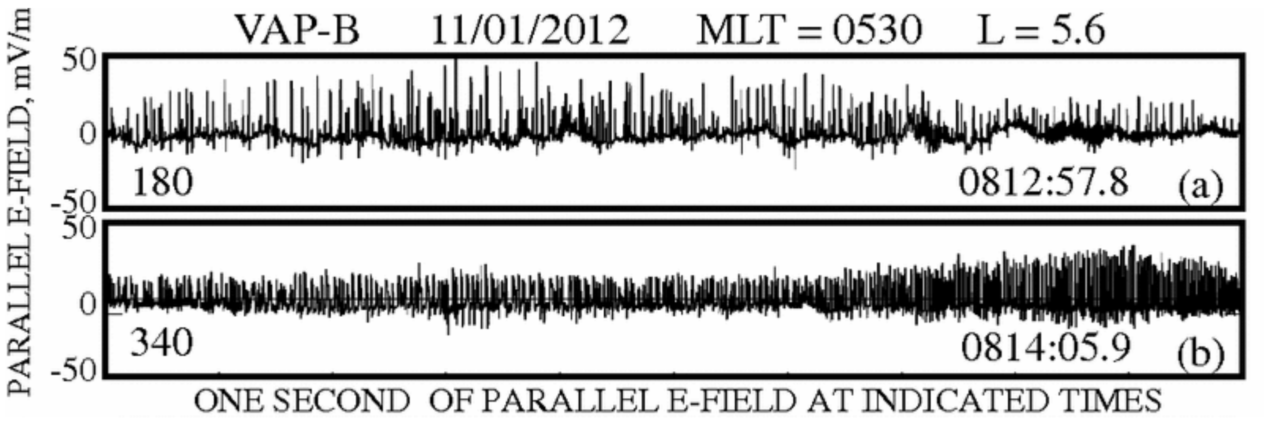
Summary

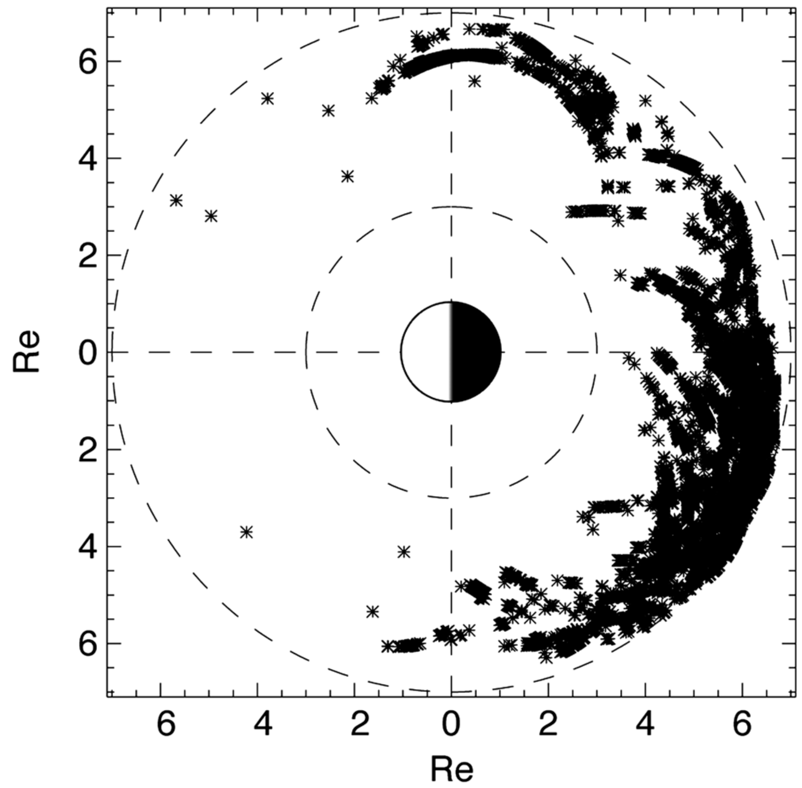
A. Discovery of localised electron structures in the Earth's magnetosphere raises several questions, e.g.,
—How are they generated and how do they dissipate?
—Are they efficient in scattering/energising electrons in the Earth's magnetosphere?

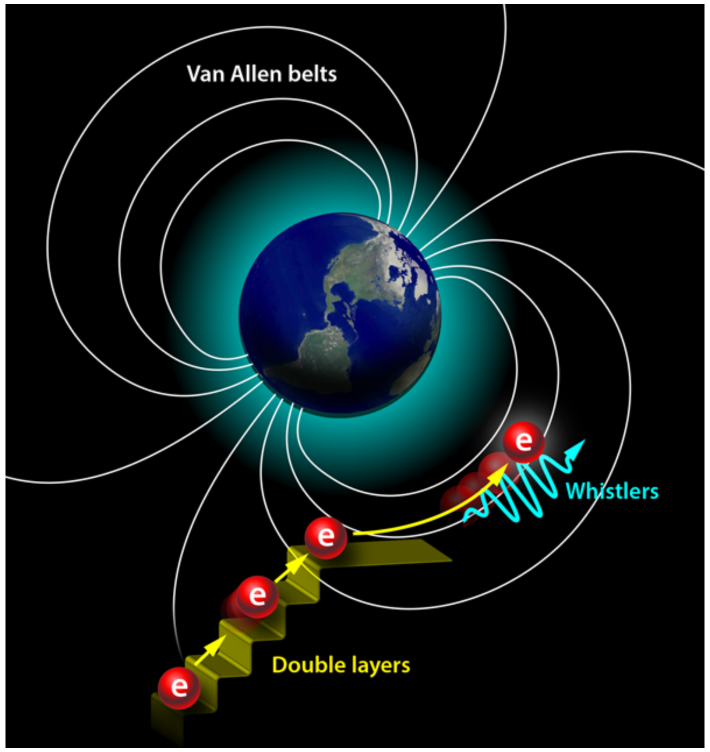
B. Since wave-particle interactions of nonlinear phase holes can NOT be treated with quasi-linear theory we derive diffusion coefficients from a Hamiltonian
— We start from Hamilton's equation to compute changes in the first adiabatic invariant.
— We incorporate a finite correlation time for the nonlinear electrostatic structures. The ratio of the linear transit time compared to the correlation time determines the strength of the wave-particle interaction.

C. The effect of finite correlation time results in enhanced diffusion of electrons with large pitch-angles (i.e. longer transit times).
—We recover the diffusion coefficients derived by Vasko et al. (2017) in the limit where the transit time is much smaller than the correlation time.

Motivation: Discovery of localised nonlinear phase space structures in the radiation belts







$$\delta E \sim 10 - 50 \text{ mV/m}$$

Mozer & al, *PRL*, **111**, (2013)
Malaspina & al, *GRL*, **41**, (2014)
Vasko & al, *JGR*, **122**, (2016)

What is the role of nonlinear phase space structures w/ large parallel electric fields in energising electrons in the Earth's magnetosphere?

Theoretical analysis

Hamiltonian for the interaction of a particle with an electrostatic phase space hole

$$H = \mu\Omega + \frac{P_z^2}{2m} + q\tilde{V}_0 \exp\left[-\frac{1}{2}\left(\frac{z - v_\phi t}{d_{||}}\right)^2 - \left(\frac{r^2+\rho^2}{2d_\perp^2}\right) + \frac{r\rho}{d_\perp^2}\cos(\theta_g)\right]. \quad (1)$$

After use of the modified Bessel function identity $e^{r\cos(\theta)} = \sum_{n=-\infty}^{\infty} I_n(r) e^{-in\theta}$ we write the Hamiltonian as:

$$H = \mu\Omega + \frac{P_z^2}{2m} + q\tilde{V}_0 \exp\left[-\frac{1}{2}\left(\frac{z - v_\phi t}{d_{||}}\right)^2\right] \sum_{n=-\infty}^{+\infty} A_n e^{-in\theta_g}, \quad (2)$$

in which the function A_n is defined as:

$$A_n = \exp\left[-\left(\frac{r^2+\rho^2}{2d_\perp^2}\right)\right] I_n\left(\frac{r\rho}{d_\perp^2}\right), \quad (3)$$

and depends on the Larmor radius ρ and the gyro-center r. The equation of motion of μ can then be written as:

$$\dot{\mu} = -\frac{\partial H}{\partial \theta_g} \quad (4)$$

$$= -iq\tilde{V}_0 \exp\left[-\frac{1}{2}\left(\frac{z - v_\phi t}{d_{||}}\right)^2\right] \sum_{n=-\infty}^{+\infty} n A_n e^{-in\theta_g} \quad (5)$$

Derivation of the diffusion coefficient: Part 1

In order to derive a diffusion coefficient we use the recipe that was outlined by Taylor (1922) almost 100 years ago for the passive transport of particle in fluid turbulence. We first compute the dispersion in the first adiabatic invariant as follow:

$$\Delta\mu\Delta\mu = 2\int_0^t dt_1 \int_0^t dt_2 \dot{\mu}(t_1)\dot{\mu}(t_2). \tag{6}$$

Since we are interested with the statistical effects of phase space holes on electrons we average over an ensemble of particles uniformly distributed along the gyrophase. With the unperturbed orbit approximation, i.e. $z = v_{\|}t$ and $\theta_g = \Omega t + \varphi_0$, valid as long as $\tau_b \gg \tau_L$, the correlation of two perturbations in the first adiabatic invariant can be written as:

$$\dot{\mu}(t_1)\dot{\mu}(t_2) = q^2\tilde{V_0}(t_1)\tilde{V_0}(t_2)\exp\left[-\frac{1}{2}\left(\frac{v_{\|}-v_\phi}{d_{\|}}\right)^2(t_1^2+t_2^2)\right]\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} nmA_nA_m e^{-in(\Omega t_1+\varphi_0)+im(\Omega t_2+\varphi_0)} \tag{7}$$

If we now integrate over the initial phase φ_0 we find the ensemble average product:

$$\langle\dot{\mu}(t_1)\dot{\mu}(t_2)\rangle_{\varphi_0} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi_0\ \dot{\mu}(t_1)\dot{\mu}(t_2)$$

Derivation of the diffusion coefficient: Part 2

We also assume, through stationarity, that the covariance of the fluctuations can be written in the form:

$$\langle \tilde{V}_0(t_1)\tilde{V}_0(t_2)\rangle_{\tilde{V}} = \langle V^2\rangle \exp(-|t_1 - t_2|/\tau_c), \tag{9}$$

in which τ_c represents the decorrelation of the electron hole fluctuations and $\langle V^2\rangle$ the variance. Thus, the integral we want to solve can be written as:

$$\langle \Delta\mu\Delta\mu\rangle_{\tilde{V},\varphi_0} = 2q^2\langle V^2\rangle \int_0^t dt_1 \int_0^t dt_2 \exp\left[-\frac{1}{2}\left(\frac{v_\parallel - v_\phi}{d_\parallel}\right)^2 (t_1^2 + t_2^2)\right] \exp(-|t_1 - t_2|/\tau_c) \sum_{n=-\infty}^{+\infty} n^2 A_n^2 e^{in\Omega(t_2-t_1)}$$

After some algebra we find:

$$\boxed{\langle \Delta\mu\Delta\mu\rangle_{\tilde{V},\varphi_0} \quad = \quad 4\pi q^2\langle V^2\rangle\tau_L^2 \sum_{n=1}^{\infty} n^2\Lambda_n^2 \exp\left(-n^2\Omega^2\tau_L^2 + \tau_L^2/\tau_c^2\right) F(\Omega,\tau_L,\tau_c,n) \quad (14)}$$

The above expression follows the integration over the gyrocentre r, and is written in terms of the function Λ_n, defined as

$$\Lambda_n = 2\exp\left(-\frac{\rho^2}{d_\perp^2}\right) \int_0^\infty x e^{-x^2} I_n^2\left(\frac{\rho x}{d_\perp}\right) dx, \tag{15}$$

Comparison with Vasko et al. (2017) results

Limit for $\tau_L/\tau_c \longrightarrow 0$ and finite $\Omega\tau_L$

The dispersion derived by (Vasko et al., 2017) for energetic electrons interacting with electron holes can nonetheless be recovered by setting $\tau_L/\tau_c \longrightarrow 0$, i.e.,

$$\lim_{\frac{\tau_L}{\tau_c}\longrightarrow 0} \langle \Delta\mu\Delta\mu\rangle_{\tilde{V},\varphi_0} = 4\pi q^2 \langle V^2\rangle \tau_L^2 \sum_{n=1}^{\infty} n^2 \Lambda_n^2 \exp\left(-n^2\Omega^2\tau_L^2\right) \tag{17}$$

In this limit the transit time is much smaller than the decorrelation time and the electron samples a phase space hole so fast that it sees no net changes of the electric field amplitude. For such an instance, and as pointed out by Vasko et al. (2017), perpendicular energisation only occurs for cyclotron resonant particles ($n\Omega\tau_L \simeq 1$) since Landau resonant particles with $n = 0$ only gain energy along the local mean field direction. However, when we account for a small, yet finite transit time with respect to the decorrelation time, an additional energy exchange mechanism for non-resonant particles appears.

Comparison with Vasko et al. (2017) results

Comparison of diffusion coefficients between $\tau_c \gg 1$ and $\Omega\tau_c \simeq 1$

In this section we quantify the effect of finite correlation time τ_c on the diffusion of particles that can be well-approximated by unperturbed orbits. For this reason, we only present figures for electrons with energies in the $10 - 100$ keV range and pitch-angle $\alpha < 70^0$ for which $\tau_L \ll \tau_b$. We now define the ratio of diffusion coefficients for comparison between Equations (14) and (18) as:

$$\boxed{\begin{aligned} R &= \frac{\langle \Delta\mu\Delta\mu\rangle_{\tilde{V},\varphi_0}}{\lim_{\frac{\tau_L}{\tau_c}\longrightarrow 0}\langle \Delta\mu\Delta\mu\rangle_{\tilde{V},\varphi_0}} \\ &= \frac{\sum_{n=1}^{\infty} n^2\Lambda_n^2 \exp\left(-n^2\Omega^2\tau_L^2 + \tau_L^2/\tau_c^2\right) F(\Omega,\tau_L,\tau_c,n)}{\sum_{n=1}^{\infty} n^2\Lambda_n^2 \exp\left(-n^2\Omega^2\tau_L^2\right)}. \end{aligned}} \qquad (18)$$

A ratio of $R > 1$ indicates that the effect of the finite correlation times enhances diffusion, whereas a ratio of $R < 1$ suppresses it. In Figure 2 we show the dependence of the parameter R as a function of the energy W and parametrised for pitch-angles $\alpha = [10^0, 30^0, 50^0, 70^0]$ and normalised decorrelation times $\tau_c = [0.75, 1, 2, 10]$. We note that the effect of finite correlation becomes important when $\Omega\tau_c \simeq 1$. For $\Omega\tau_c \gg 1$, the convergence to unity indicate that the diffusion is to a few percents identical to that of Vasko et al. (2017). For pitch-angles $\alpha \leq 40^0$, the finite correlation time suppresses diffusion by a factor of as large as 30%. On the other hand, for pitch-angles $\alpha \geq 50^0$, $\Omega\tau_c \simeq 1$ enhances diffusion by a comparable factor of 30%. In Figure 3, we plot the same ratio

Results

Orbits that can be described as unperturbed require transit time less than trapping time.

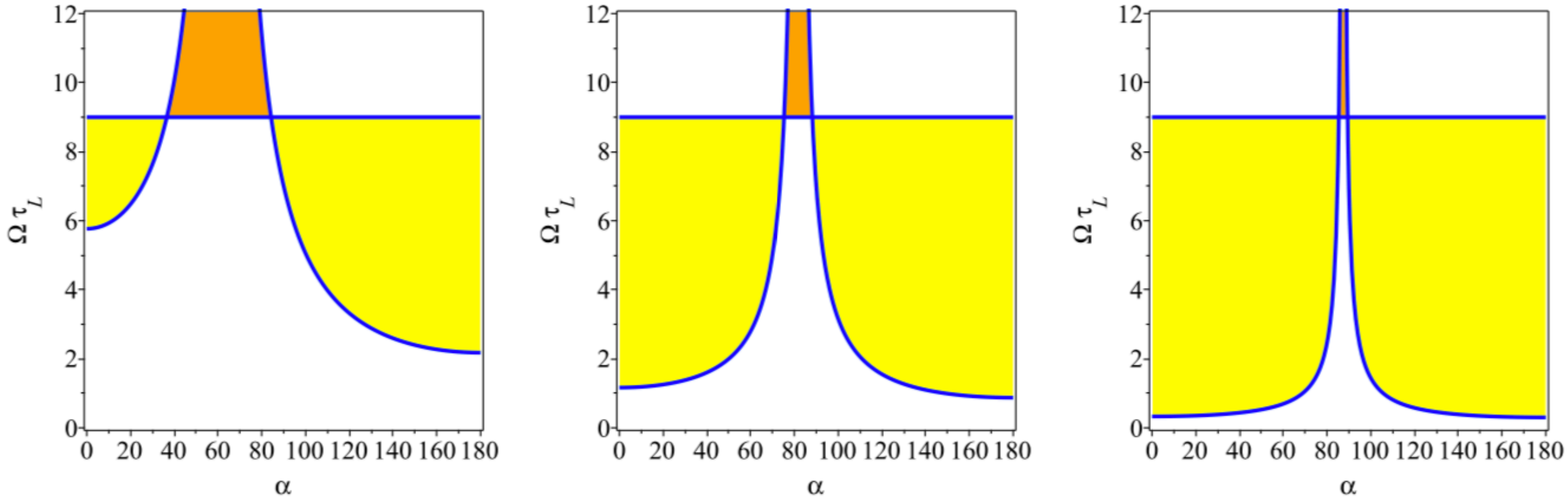


Figure 1. Normalised transit time $\Omega\tau_L = \frac{\Omega d_{\|}}{v_{\|} - v_{\Phi}}$ as a function of the pitch-angle α for, starting from the left most panel, fixed energies $W = [0.1, 1, 10]$ keV. The normalised trapping time for an electron interacting with a phase space hole of electric field amplitude $E \simeq 30$ mV/m and scale $d \simeq 0.7$ km is of the order of $\Omega\tau_b \simeq 9$. The yellow (orange) shaded regions represents the parameter space for which transit times are smaller (larger) than bounce time. In this study we focus solely on electrons that have transit times much smaller than the bounce time, i.e. $\Omega\tau_L \leq 1$. This constraint is made to insure that the unperturbed orbit approximation remains valid.

Ratio of diffusion coefficient: Part 1

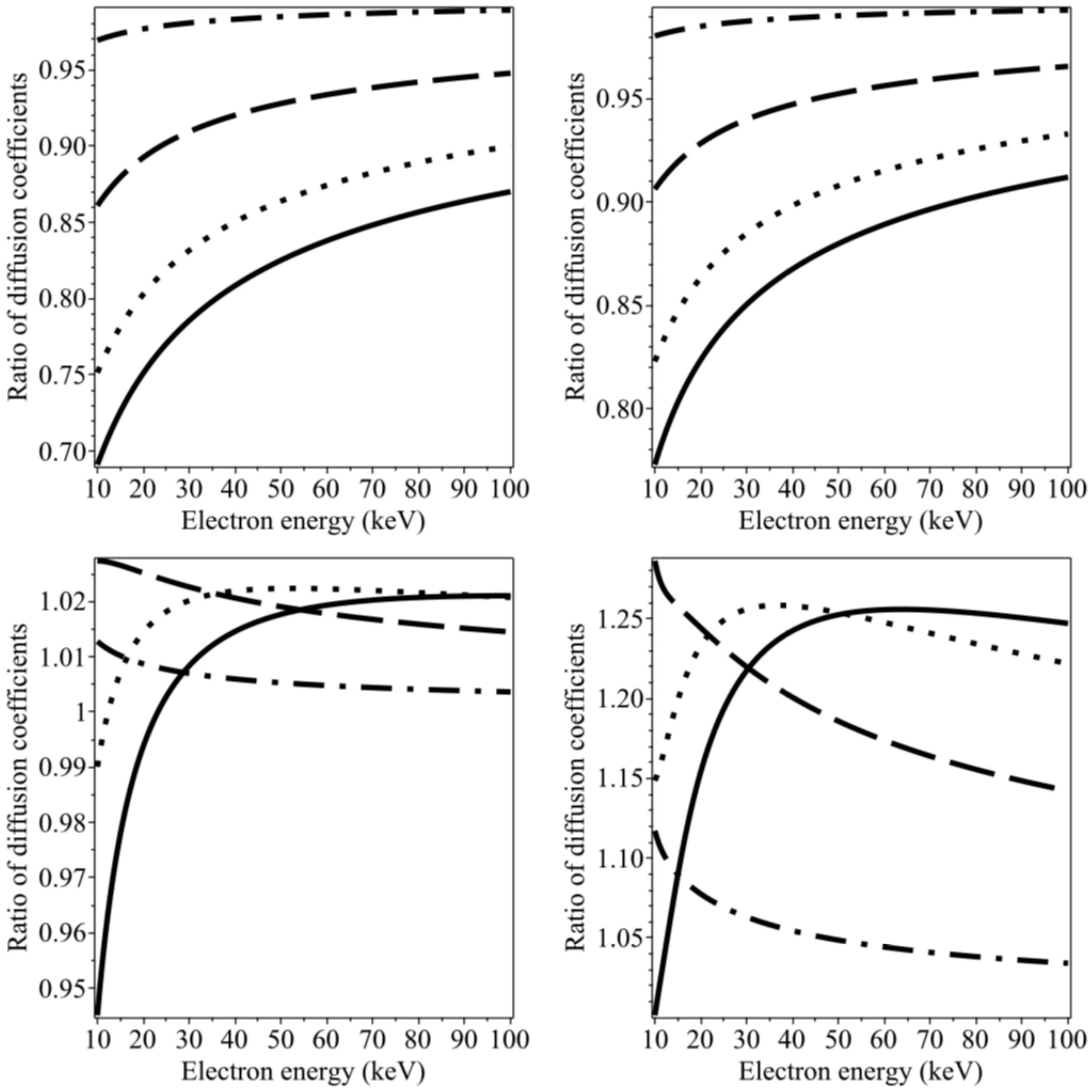


Figure 2. Ratio of diffusion coefficients with and without finite correlation time effects. The top left (right) panel shows the energy dependence of the coefficient R for electrons with 10 (30) degrees pitch-angle. The bottom left (right) panel shows the same coefficient but for electrons with 50 (70) degree pitch-angle. The solid, dotted, dash, and dashed-dotted curves in each panel are for normalised correlation times $\Omega_e \tau_c = [0.75, 1, 2, 10]$. Note that the scale of the y-axis in each panels differ.

Ratio of diffusion coefficient: Part 2

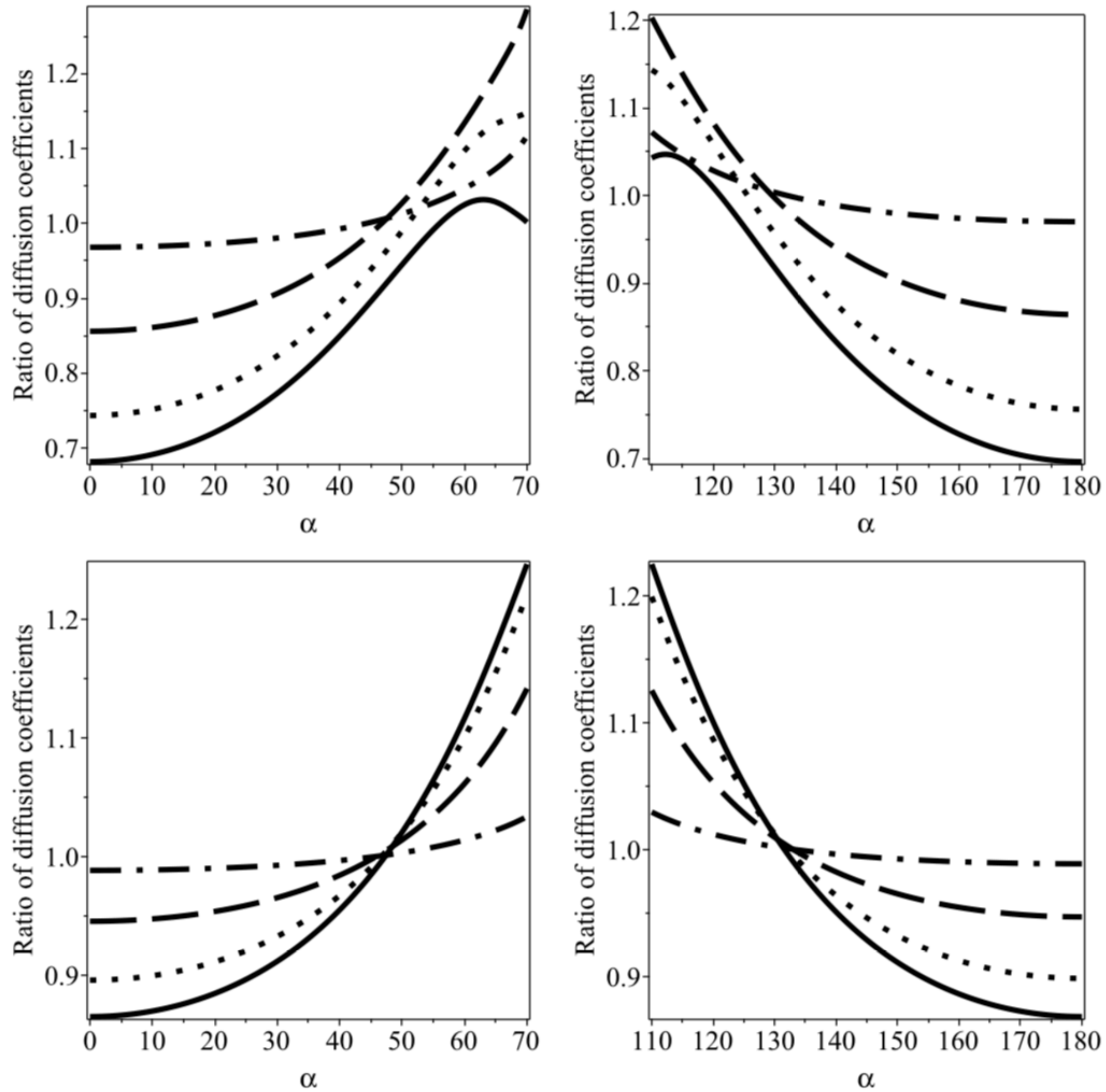


Figure 3. Ratio of diffusion coefficients with and without finite correlation time effects. The top panels show the pitch-angle dependence of the coefficient R for electrons with $W = 10$ keV. The bottom panels show the same coefficient but for electrons with $W = 100$ keV. The solid, dotted, dash, and dashed-dotted curves in each panel are for normalised correlation times $\Omega_e\tau_c = [0.75, 1, 2, 10]$. We only compute the ratio for particles with transit-times smaller than the bounce time, i.e. $\tau_L \ll \tau_b$, with the cut-off between $\alpha = 70^0$ and $\alpha = 110^0$ a results of which.

Conclusion/Take away

A. Using a Hamiltonian formalism we derived a diffusion coefficient while taking into account the finite correlation time of the electric fields of nonlinear phase space structures——> electrons can sample decorrelating/growing fields.

B. This effect results in enhanced diffusion for e- that have longer transit time (i.e. large pitch-angles) and reduced diffusion for e- that have short transit time (small pitch-angles).

C. The diffusion of 10-100 keV electrons caused by phase-holes is comparable in size to that by whistlers. Our results indicate that the associated diffusion coefficients need to be incorporated into global models.