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Highly accurate analytical footprint model for general stratification of the atmosphere

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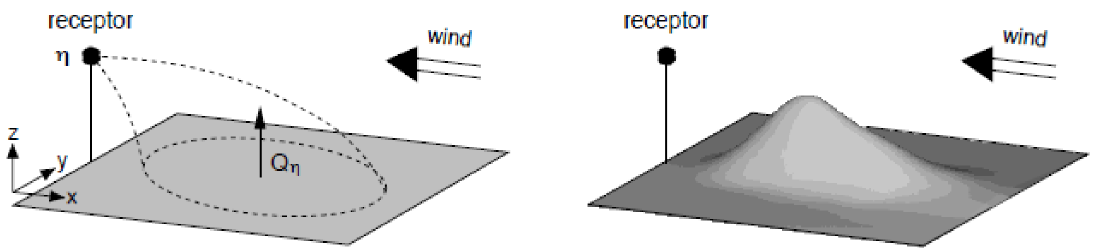
Footprint definition

The notion of footprint is used to describe the spatial extent and position of the surface area that is contributing to a turbulent **concentration** or **flux measurement** at a specific height for specific atmospheric conditions and surface characteristics.

Footprint relatively to the measurement of
- a passive scalar (i.e. : CO_2 or pollutant concentration)
- a vertical flux (heat, humidity, pollutant …)



Kljun, 2002

$$\eta(x_m, y_m, z_m) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q(x', y', z=0) \boxed{f_\eta(x_m - x', y_m - y', z_m)} \, dx' dy'$$

or

$\chi(x_m, y_m, z_m)$

scalar (i.e.: concentration)

$\overline{w'\chi'}$

Vertical turbulent flux

The footprint function corresponds to the convolution kernel

It then appears that $f_\eta(x_m, y_m, z_m)$ is equal to the **response** $\eta(x_m, y_m, z_m)$ **at height** z_m **to a unitary point source at ground at (0,0,0)**

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Existing footprint models

The footprint calculation amounts to solve a 3D turbulent dispersion problem in the atmospheric boundary layer.

Often, the calculations are restricted to the surface layer.

Lagrangian stochastic particle dispersion models

(e.g. *Hsieh, 2000, Kljun 2004, 2015*)

- \+ General purpose
- \+ May be applied outside the surface layer, below the entrainment layer
- − High computing time

Surrogate models have been proposed (analytical parameterizations)

- \+ Very fast
- − Biased results (fitting errors)

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Existing footprint models

Tools based on the analytical solution of a (close) advection-diffusion problem

- Variable separation

$$f_\eta(x,y,z_m)=\chi_y(x,y)\,\overline{f_\eta}^{\,y}(x,z_m)$$

Gaussian crosswind dispersion

2D crosswind-integrated footprint

- 1st order turbulence closure : K-theory

$$\overline{w'\chi'}\approx -K(z)\frac{\partial \chi}{\partial z}$$

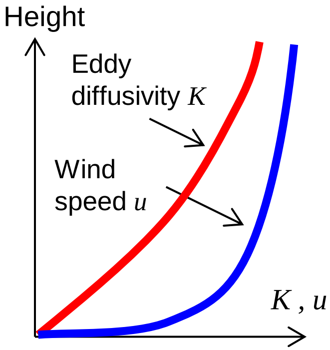
Eddy diffusivity (height-dependent)

- Analytical modeling of the associated 2D advection-diffusion process :

$$u(z)\frac{\partial \chi}{\partial x}=\frac{\partial}{\partial z}\left(K(z)\frac{\partial \chi}{\partial z}\right)$$

Wind speed

Eddy diffusivity



- + Very fast
- − Restricted to the surface layer
- − Considered analytical models neglect longitudinal dispersion
- − The underlying analytical models are restricted to **power-law** profiles of u and K

New (semi-)analytic model

New method for the calculation of the crosswind-integrated footprint $\overline{f_\eta}^{y}(x, z_m)$

Semi-analytic method to solve the advection-diffusion equation : $$u(z)\frac{\partial \chi}{\partial x} = \frac{\partial}{\partial z}\left(K(z)\frac{\partial \chi}{\partial z}\right)$$

based on an **optimal adjustment of the relevant parameters with the actual turbulent profiles.**
The adjustment is realized by :

- performing a **Liouville transformation**, which gives rise to a new parameter agglomerating $u(z)$ and $K(z)$, namely the atmosphere **reluctance**: $b(z) = \sqrt{u(z)K(z)}$
- splitting the height range in due number of sublayers and performing a **(extended-type) power-law** fit for the reluctance in each sublayer
- solving the advection-diffusion equation by adapting the **quadrupole** method (e.g. *Maillet, 2000, Krapez, 2014*) to the present **extended-type power-law profiles**

+ Very fast (< 1s)
+ Perfect adjustment with e.g. Monin-Obukhov profiles (or whatever else profiles)
+ Not restricted to the surface layer (insofar as the input profiles $u(z)$ and $K(z)$ conform the real ones)

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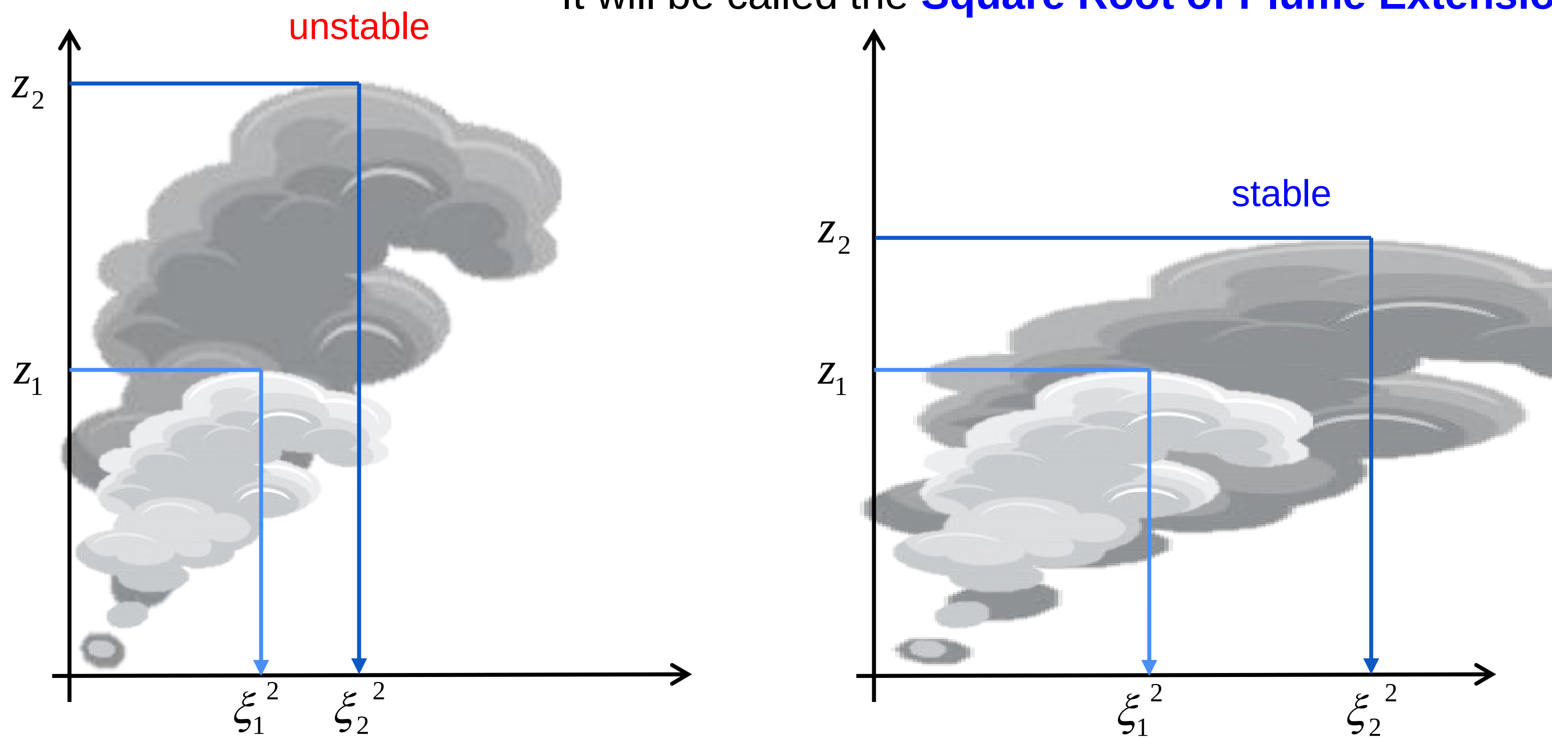
New paradigm : consider a change of space (Liouville transformation)

The Liouville Transformation consists of a change of the independent variable $z \to \xi$ (see e.g. *Krapez, 2016*)

$$\xi = \xi(z) := \int_0^z \sqrt{\frac{u(z')}{K(z')}} dz'$$

The new independent variable $\xi(z)$ can be interpreted as the square root of the longitudinal distance covered by a plume when reaching height z.

It will be called the **Square Root of Plume Extension (SRPE)**



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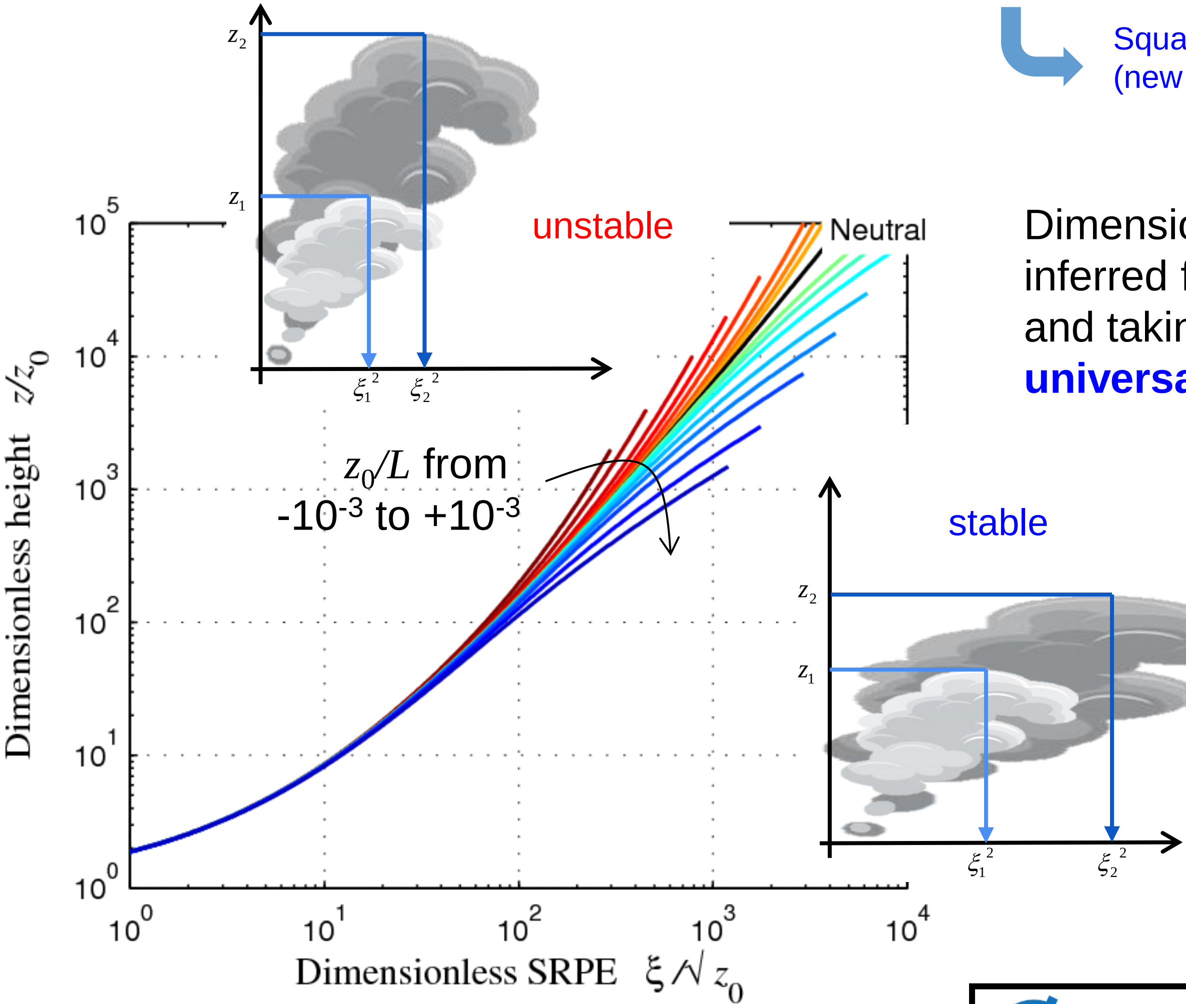
New paradigm : consider a change of space (Liouville transformation)

Liouville Transformation : $z \to \xi\,;\qquad \xi = \xi(z) := \int_0^z \sqrt{\frac{u(z')}{K(z')}}\,dz'$

Square Root of Plume Extension - SRPE (new independent variable)



Dimensionless profiles of **SRPE** as inferred from **MO similarity theory**, and taking **Businger/Högström universal similarity functions**

Horizontal plume extension is higher in stable conditions

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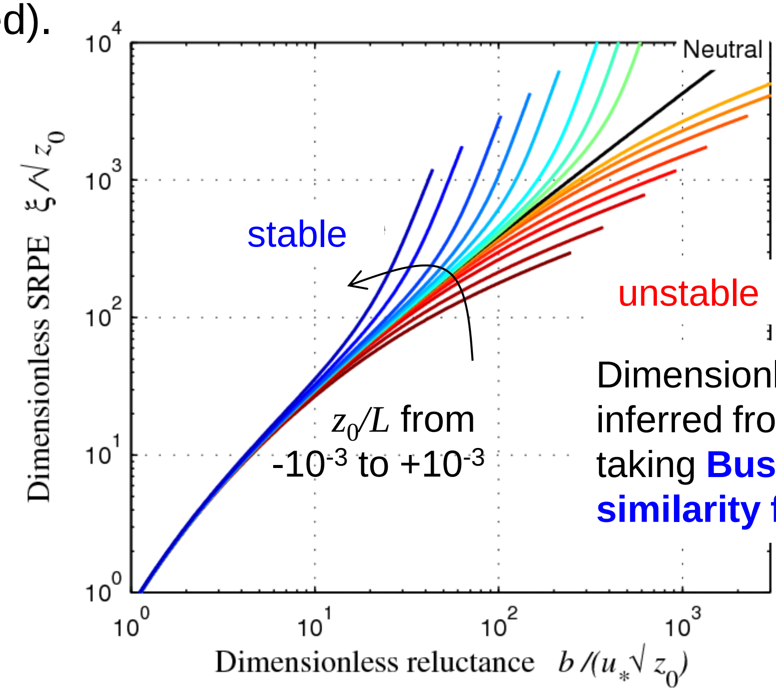
New paradigm : consider a change of space (Liouville transformation)

In the Liouville space, the advection-diffusion equation becomes:

$$b(\xi)\frac{\partial \chi}{\partial x} = \frac{\partial}{\partial z}\left(b(\xi)\frac{\partial \chi}{\partial z}\right) \quad \text{with} \quad b(\xi) = \sqrt{u(\xi)K(\xi)}$$

Instead of **two profiles** (wind speed and eddy diffusivity), it now features **only one profile**: $b(\xi)$ which has been called the **atmosphere reluctance** (namely **inertia of the atmosphere to concentration changes**).

All the concentration and flux features (in particular the footprints) are entirely determined by the single profile of reluctance $b(\xi)$ which is therefore a parameter of uttermost importance (yet overlooked).



Dimensionless profiles of **reluctance** as inferred from **MO similarity theory**, and taking **Businger/Högström universal similarity functions**

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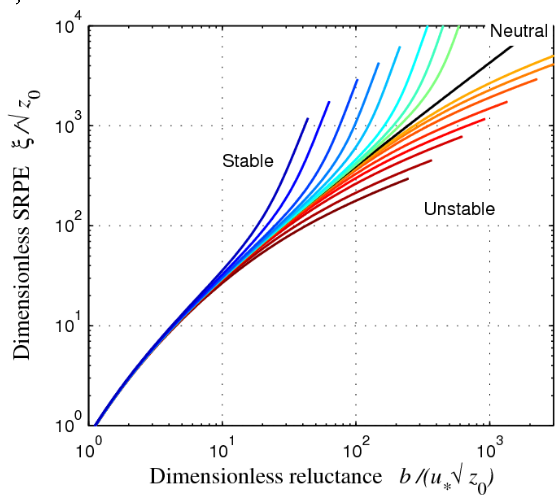
New paradigm : consider a change of space (Liouville transformation)

The advection-diffusion equation $b(\xi)\chi_{,x} = \left(b(\xi)\chi_{,z}\right)_{,z}$ has an analytical solution for power-law profiles of reluctance.

However, the **MO reluctance profiles** are not power-law profiles. They are actually unlikely to be analytically solvable profiles.



Profiles of reluctance deriving from the MO theory (or from parametrizations extending beyond the surface layer) will be **piecewise approximated by solvable profiles of high flexibility.**

It was shown that the family of **extended-power-law profiles** defined by

$$b(\xi) = \hat{\xi}\left[A_B \hat{\xi}^{\nu} + A_D \hat{\xi}^{-\nu}\right]^2 \quad ; \quad \hat{\xi} = \xi + \xi_c$$

are **analytically solvable and flexible (four adjustable free parameters) while suitable for high-range variations.**

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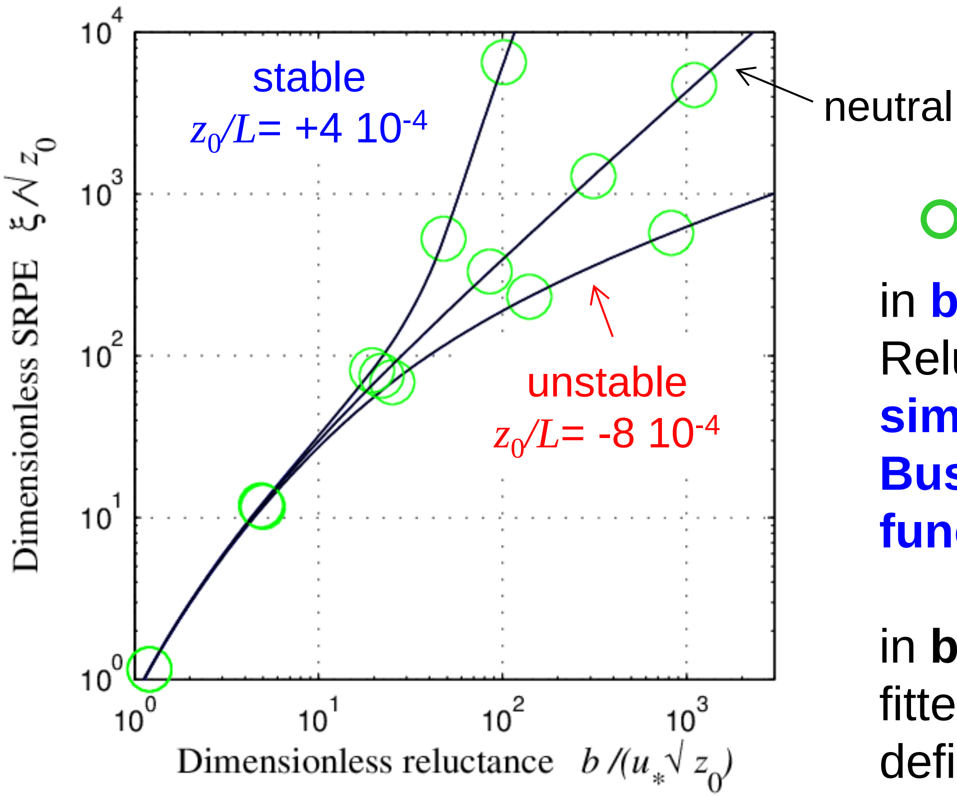
Gooness of fit



: discretization nodes

in **blue**:
Reluctance profiles derived from **MO similarity theory**, while taking **Businger/Högström universal similarity functions**

in **black** (perfectly overlapping):
fitted **extended-power-law profiles** defined by :

$$b(\xi) = \hat{\xi}\left[A_B \hat{\xi}^{\nu} + A_D \hat{\xi}^{-\nu}\right]^2 \quad ; \quad \hat{\xi} = \xi + \xi_c$$

These profiles can be considered as **solvable splines**.
With the chosen discretization density, the fitting error is less than 0.1%
Their application is **not restricted to the surface layer**.

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Semi-analytic calculation

Quadrupole method (or **Transfer Matrix** method) in the Laplace domain (the classical case of homogeneous sublayers has been described in, e.g. *Maillet, 2000, Krapez, 2014*)

Quadrupole matrices have been defined for the present case of **extended-power-law profiles.**

Simple algebra with these matrices allows computing the concentration and the flux at any height of the composite boundary layer (see e.g. *Krapez 2014, 2016*).

Numerical inverse Laplace transform to get the concentration and the flux vs. downwind distance (e.g. *Krapez 2014)*

Fast (< 1s) and highly accurate semi-analytic solution for concentration, flux, and footprint as a by-product

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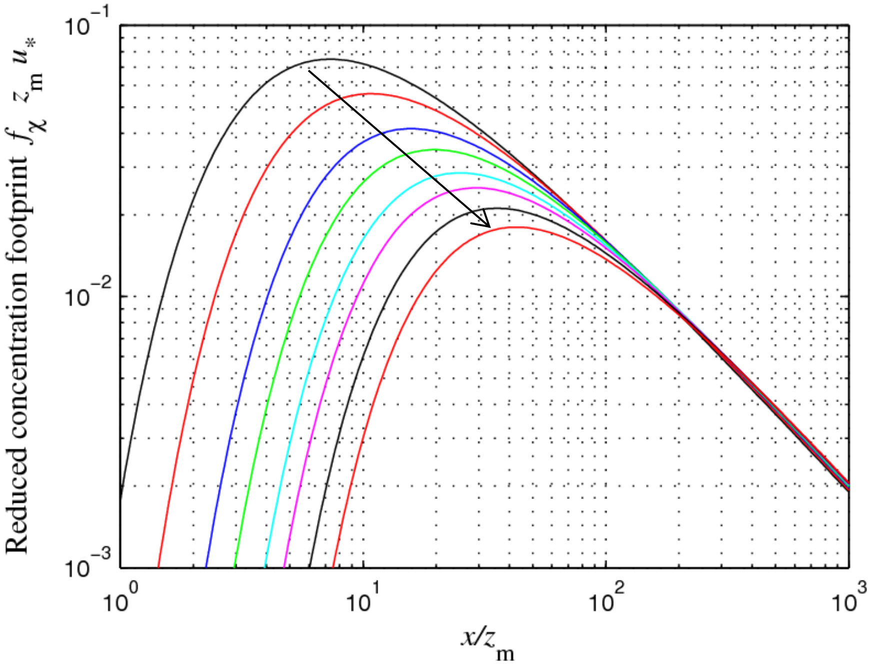
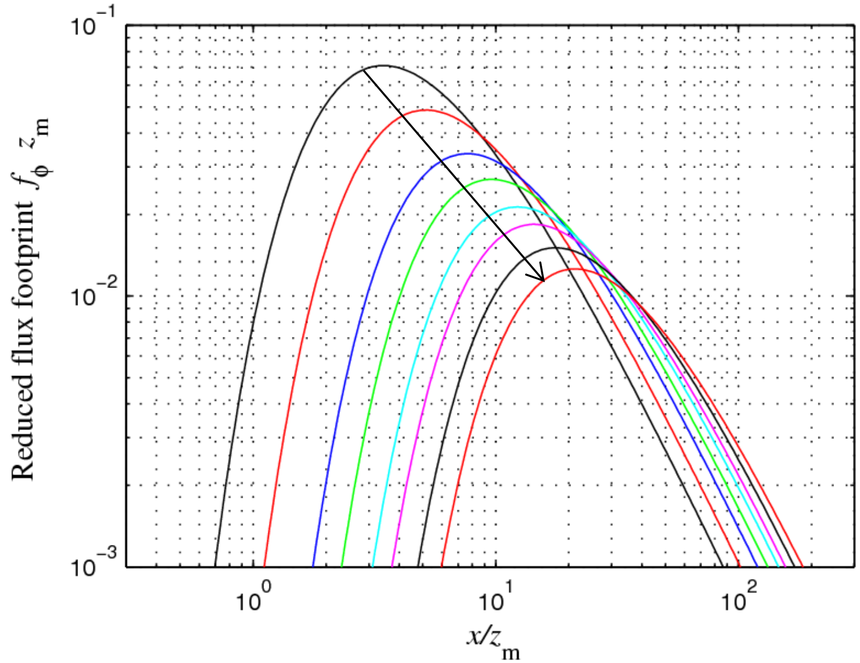
Example for the neutral case

Concentration footprint (crosswind integrated)

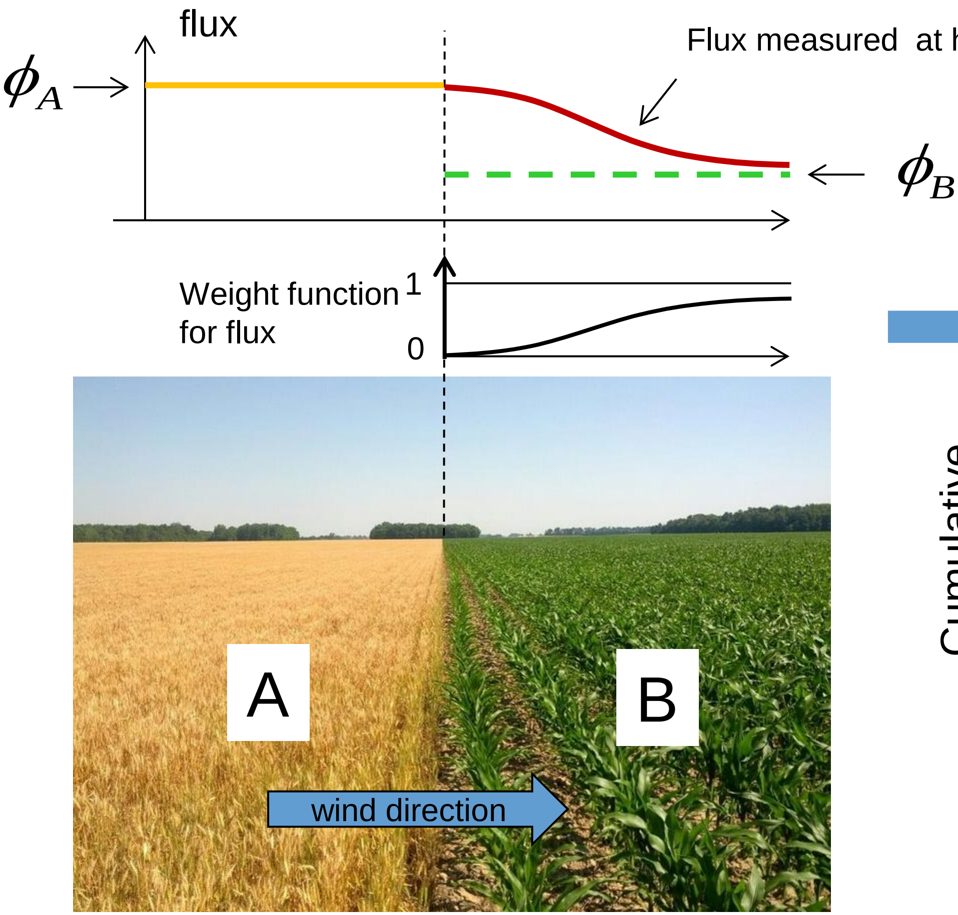


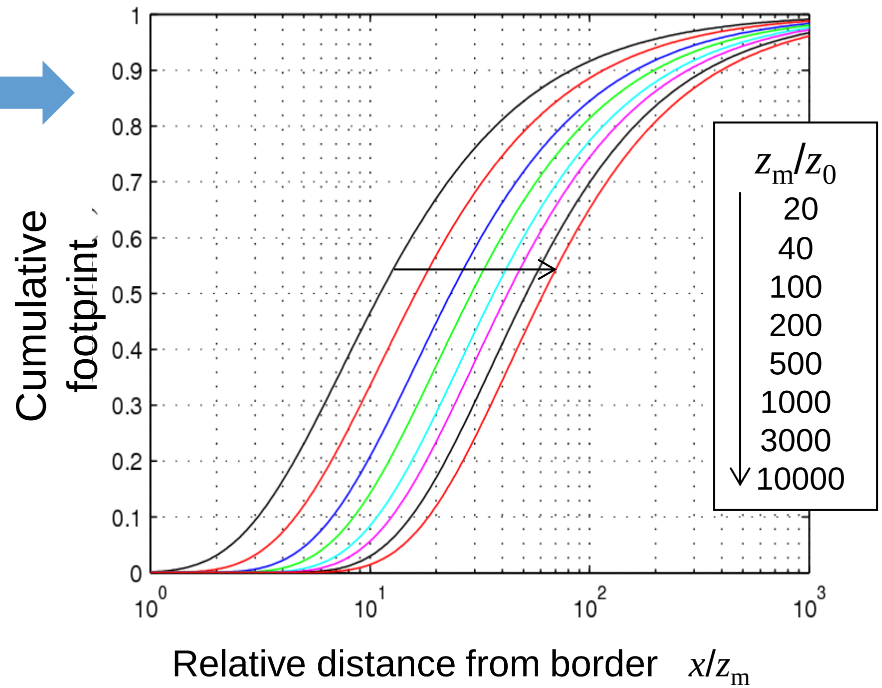
Flux footprint (crosswind integrated)



Normalized measurement height z_m/z_0 = 20, 40, 100, 200, 500, 1000, 3000, 10000

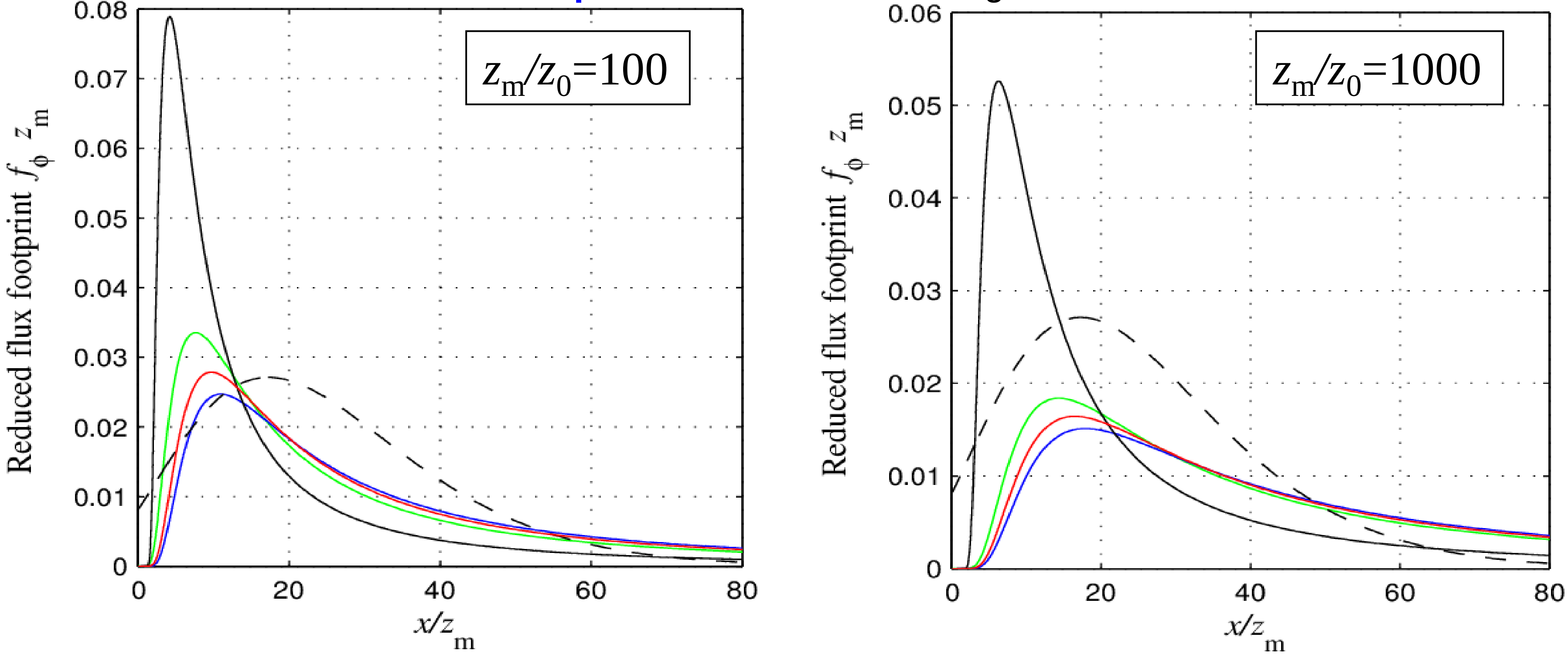
Fetch in neutral case





Comparaison with other models (neutral case)

Flux footprint for two different heights of the sensor



- *Kljun, 2015* (Lagrangian+metamodel)
- *Kljun, 2004* (Lagrangian+metamodel, z_0=0.04m)

with longitudinal turbulence

- *Hsieh, 2000* (Lagrangian+metamodel)
- *Kormann, 2001* (power-law profiles for u and K)
- present model (quasi exact diffusion model)

without longitudinal turbulence

Large dispersion of results. Might be partly explained by the longitudinal turbulence $\overline{u'\chi'}$

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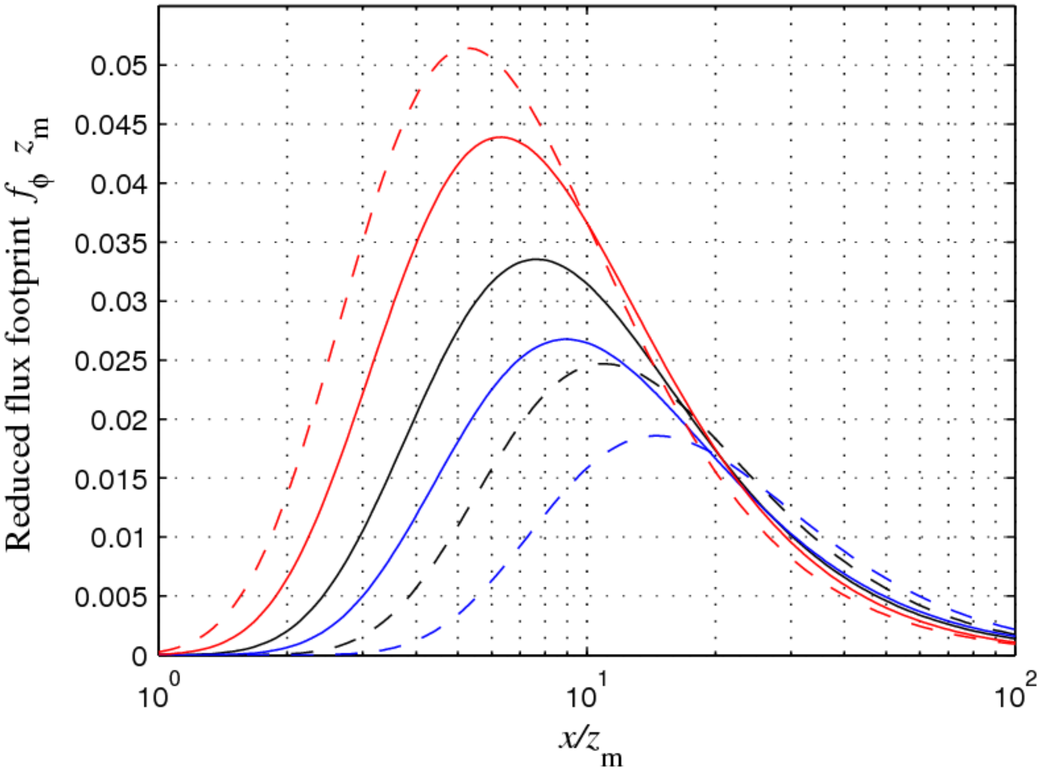
Influence of stability. Comparaison with a Lagrangian model (*Hsieh, 2000*)

Flux footprint



L : Obukhov length

z_0/L

- -0.0008 (unstable)
- 0 (neutral)
- +0.0004 (stable)

dashed lines : *Hsieh, 2000*

Illustration for z_0=0.04m, z_m=4m
L=-50m (unstable), inf. (neutral), +100m (stable)

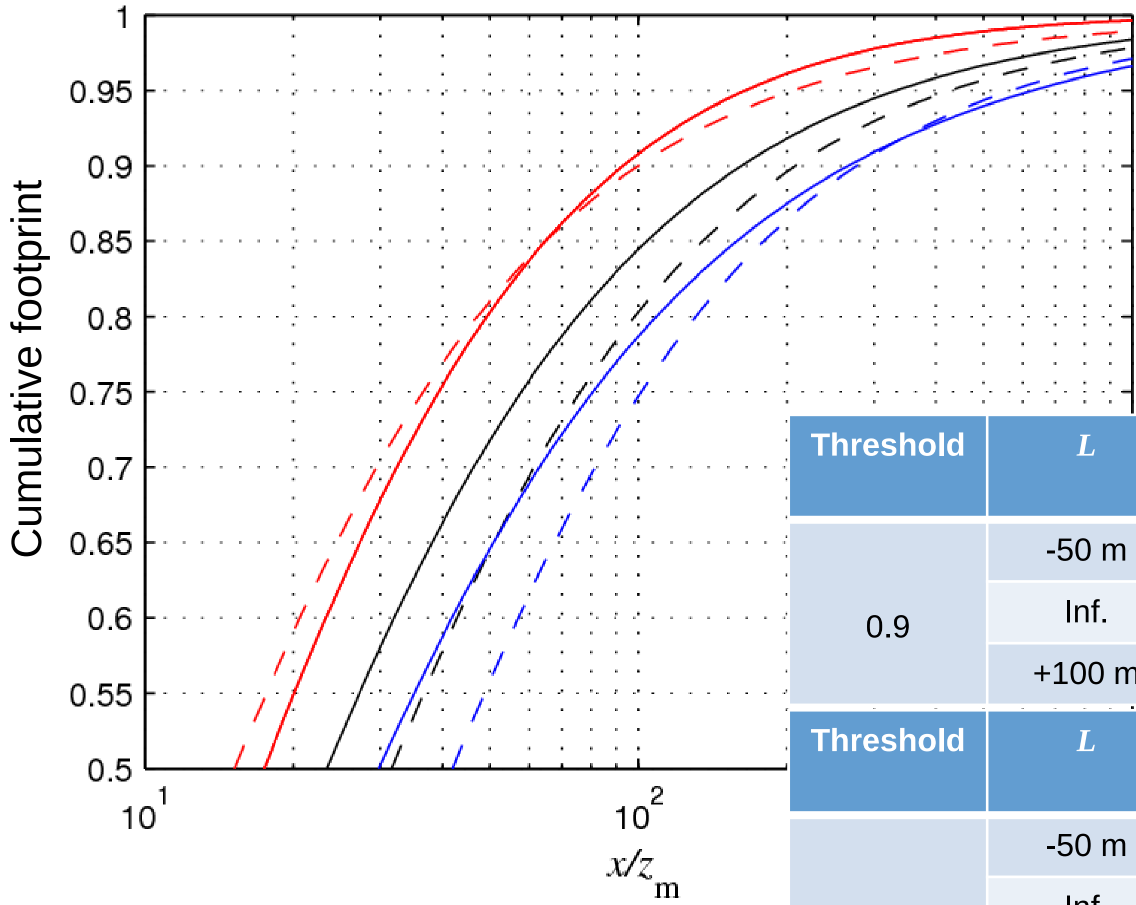
Position of maximum x/z_m

L	Present method	Hsieh, 2000	Difference
-50 m	6.3	5.2	-17%
Inf.	7.7	11.0	+43%
+100 m	9.0	14.5	+62%

The Lagrangian model (*Hsieh, 2000*) gives a maximum that is
- closer in the case of **unstable** condition
- further downwind in the case of **neutral** or **stable** condition

Influence of stability. Comparaison with a Lagrangian model (*Hsieh, 2000*)

Cumulative flux footprint



L : Obukhov length

z_0/L

-0.0008 (unstable)

0 (neutral)

+0.0004 (stable)

dashed lines : *Hsieh, 2000*

Illustration for z_0=0.04m, z_m=4m
L=-50m (unstable), inf. (neutral), +100m (stable)

Threshold	L	Present method	Hsieh 2000	DIfference
0.9	-50 m	93.0	99.9	+7%
	Inf.	161	208	+29%
	+100 m	266	276	+4%

Threshold	L	Present method	Hsieh 2000	DIfference
0.75	-50 m	39.3	36.6	-7%
	Inf.	57.8	76.2	+32%
	+100 m	80.8	101.2	+25%

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Discussion and outlook

The current model allows to solve in a **short time (<1s)** and with **great accuracy** the equation of the mean (turbulent) concentration provided that:

- parameterizations of the vertical profiles of wind velocity $u(z)$ and $K(z) := -\overline{w'\chi'}/(\partial\chi/\partial z)$ are well representative of the turbulent structure of the atmosphere
- the term of longitudinal dispersion $\overline{u'\chi'}$ is negligeable when compared to the advection term

- suitable for the **surface layer** by using MO similarity functions
- may be extended **beyond the surface layer** if due profiles of $u(z)$ and $K(z)$ are available

- the present model **outperforms classical « analytical » footprint models** (e.g. *Korman, 2001*) and LPD models that assume the same approximation (e.g. *Hsieh, 2000*)
- may be used for pollutant-dispersion modeling as a tradeoff between Gaussian models and more sophisticated (albeit more time consuming) models (e.g. *Sarrat 2017*)
- turbulent longitudinal dispersion will be added in the model in the next future.

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