# Modular Korteweg-de Vries Equation


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# Korteweg-de Vries Equation (1895)

Diederik Johannes Korteweg
(1848-1941)

Gustav de Vries
(1866-1934 )

Discovered for water waves, KdV equation describes weakly nonlinear and weakly dispersive waves in many physical systems

$$\boxed{\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0}$$

Canonical form for unidirectional propagation in reference system x – c t

# Modified Korteweg-de Vries Equation

$$\boxed{\frac{\partial u}{\partial t} \pm 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0}$$

- is well-known in nonlinear mathematics (fully integrable model),

- less in nonlinear physics

**T.L. Perelman, A.Kh. Fridman, M.M. Yelyashevich**, Modified KdV equation in electro-dynamics. *Sov. Phys. JETP , 1974, vol.* 39, 643-646.

**Pelinovskii E.N., and Sokolov V.V.** Nonlinear theory for the propagation of electromagnetic waves in size-quantized films*. Radiophysics and Quantum Electronics*, 1976, vol. 19, N. 4, 378 -382.

$$v \frac{\partial \mathscr{E}}{\partial x} + \frac{\partial \mathscr{E}^3}{\partial \tau} + \frac{1}{\omega_0^2} \frac{\partial^3 \mathscr{E}}{\partial \tau^3} + \frac{2\nu}{\pi \omega_0} \int_{-\infty}^{\infty} \frac{\partial \mathscr{E}}{\partial \tau'} \frac{d\tau'}{\tau - \tau'} = 0.$$

# Plasma Waves and KdV-like equations

**Schamel, H.** A modified Korteweg-de Vries equation for ion acoustic waves due to resonant electrons. *J. Plasma Phys*. 9, 377–387 (1973)

$$\boxed{\frac{\partial u}{\partial t}+\frac{5}{2}\frac{\partial |u|^{3/2}}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0}$$

**Schamel Equation**

**Gardner equation**

$$\boxed{\frac{\partial u}{\partial t}+(\alpha u+\alpha_1 u^2)\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0}$$

**Ruderman M.S., Talipova T., Pelinovsky, E.** Dynamics of modulationally unstable ion-acoustic wave packets in plasmas with negative ions. *J. Plasma Physics,* 2008, vol. 74, No. 5, 639-656

**S.A. El-Tantawy, E.I. El-Awady, R. Schlickeiser.** Freak waves in a plasma having Cairns particles. *Astrophys Space Sci* 2015, vol. 360, 49.

**S.A. El-Tantawy.** Rogue waves in electronegative space plasmas: The link between the family of the KdV equations and the nonlinear Schrödinger equation. *Astrophys Space Sci,* 2016, vol. 361. 164

# Logarithmic KdV equation

## Granular chains with Hertzian interaction forces

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u \log|u|\right) + \frac{\partial^3 u}{\partial x^3} = 0}$$

## Gaussian soliton

$$\boxed{u(x-ct) = \exp\left[c + \frac{1}{2} - \frac{(x-ct-x_0)^2}{4}\right]}$$

**R. Carles and D. Pelinovsky.** On the orbital stability of Gaussian solitary waves in the log-KdV equation, *Nonlinearity*, 2014, vol. 27, 3185 -3202.

**E. Dumas and D.E. Pelinovsky.** Justification of the log-KdV equation in granular chains: the case of precompression, *SIAM J. Math. Anal*., 2014, vol. 46, 4075 - 4103.

**G. James and D. Pelinovsky.** Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, *Proc. Roy. Soc. A*, 2014, vol. 470, 20130465.

# Elastic Waves in Bimodular Media

$$\rho\frac{\partial^2\varepsilon}{\partial t^2}=\frac{\partial^2\sigma}{\partial x^2}+\gamma\frac{\partial^4\sigma}{\partial x^4},\qquad \varepsilon=\frac{1}{E}\left(\sigma+g\left|\sigma\right|\right).$$

**σ – stress**

**ε - deformation**

**different response to tensile and compressive stresses**

$$\boxed{\frac{\partial u}{\partial t}=\frac{\partial\,|u|}{\partial x}+\frac{\partial^3 u}{\partial x^3}}$$

***"Modular" KdV Equation***


**Rudenko O.V. Modular solitons, *Doklady Mathematics*, 2016, vol. 94, 708-711**

**Nazarov V., Kiashko S., Radostin A. Wave processes in bimodular media. *Radiophysics and Quantum Electronics*, 2016, vol. 59, 275-285**

# Modular KdV hierarchy

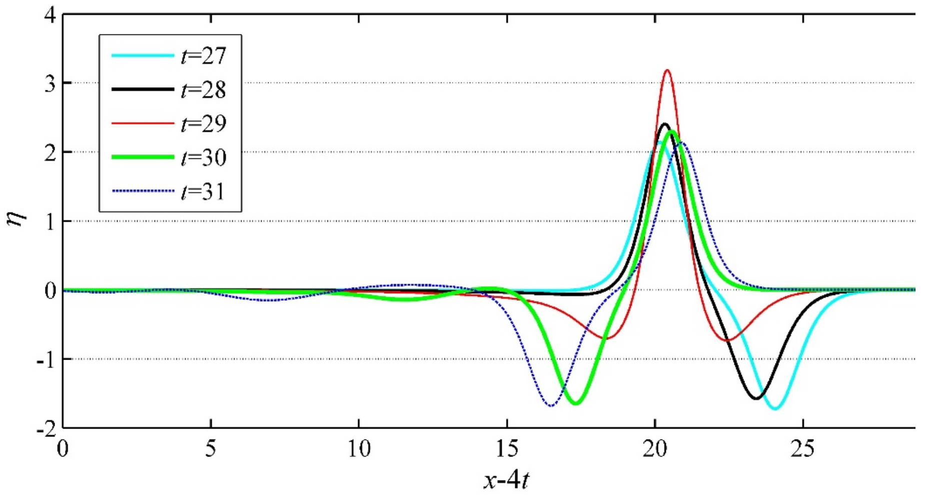
$$\boxed{\frac{\partial u}{\partial t} + 6|u|\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0}$$

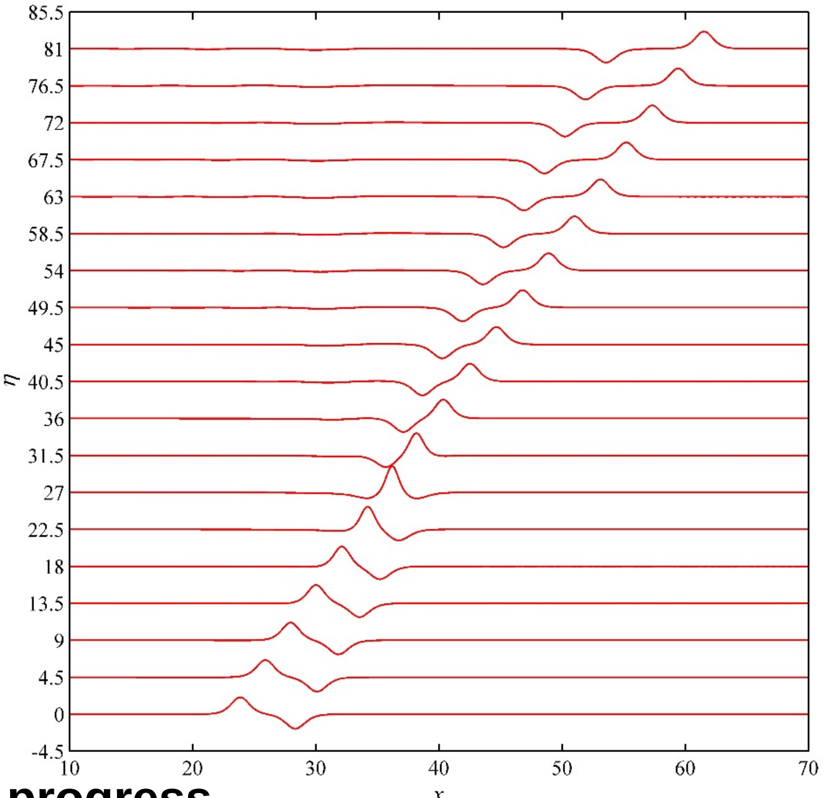
***Canonical "Modular" KdV Equation***

**Rudenko O.V. Nonlinear dynamics of quadratically cubic systems. *Physics - Uspekh*i 56 (7) (2013)**

**"Semi"-integrability**

**N-soliton solutions**




**Strong amplification – rogue wave?**

**Kokorina A., Pelinovsky E., Slunyaev A. – in progress**

# Dispersionless modular KdV Equation

$$\boxed{\frac{\partial \eta}{\partial t} + V(\eta)\frac{\partial \eta}{\partial x} = 0}$$

$$\boxed{V(\eta) = 6|\eta|}$$

$$\boxed{\eta(x,t) = F[x - V(\eta)t]}$$

**Riemann Wave**

**Wave Steepness**

**First Wave Breaking**

$$\boxed{\frac{\partial \eta}{\partial x} = \frac{dF/dx}{1 + \dfrac{dV}{dx}t}}$$

$$\boxed{T = \frac{1}{(-dV/dx)_{\max}}}$$

# Fourier spectrum of a nonlinearly deformed wave

$$\boxed{\eta(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty}\left[a_n(t)\cos(nkx) + b_n(t)\sin(nkx)\right]}$$

$$\boxed{S_n(t) = a_n + ib_n = \frac{k}{\pi}\int_0^{2\pi/k} \eta(x,t)\exp\left[inkx\right]dx}$$

**Implicit formula**

**Change of variable** $\boxed{y = x - V(\eta)t}$

$$\boxed{S_n(t) = \frac{i}{n\pi}\int_0^{2\pi/k} \frac{dF}{dy}\exp\left\{ink\left[y + V(F)t\right]\right\}dy}$$

***F(y) = A sin(ky)***

**Explicit formula**

$$\boxed{S_n(t) = \frac{iA}{n\pi}\int_0^{2\pi} \cos x \exp\left\{in\left[x + \left(6kA\,|\sin x\,|\right)t\right]\right\}dx}$$

**Final formula**

## Dispersionless Modular KdV equation

$$\boxed{\frac{\partial u}{\partial t} + |u|\frac{\partial u}{\partial x} = 0}$$

$$u(x,t) = AJ_0(Akt)\sin(kx) + A\sum_{m=0}^{\infty}\frac{1}{2m+1}J_{2m+2}[(2m+1)Akt]\sin[(2m+1)kx -$$

$$-\frac{4A}{\pi}\sum_{m=0}^{\infty}\frac{\cos[(2m+1)kx]}{2m+1}\sum_{n=0}^{\infty}J_{2n+1}[(2m+1)Akt]\left(\frac{2n+2}{(2n+2)^2-(2m+1)^2} + \frac{2n}{4n^2-(2m+1)^2}\right)$$

**Tobisch E., Pelinovsky E. Modular Hopf equation.** ***Applied Math Letters*****, 2019, 97,1-5**

## Intensive harmonic generation

## Breaking asymptotic – the same

$$u(x,t) = A\sin(kx) + tu_1(x)$$

$$u_1(x) = \frac{4A^2k}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2-4}\cos[(2m+1)kx]$$

# Universal Spectrum Asymptotics

## $S(k) \sim k^{-4/3}$

**means existence of singularity in wave shape**

$$\boxed{\eta(x,t) \sim (x - x_{br})^{1/3}}$$

**Proof:**

$$\boxed{\frac{\partial \eta}{\partial t} + V(\eta)\frac{\partial \eta}{\partial x} = 0} \quad \longrightarrow \quad \boxed{\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = 0}$$

**Or in equivalent form** $\boxed{\dfrac{dx}{dt} = V}$ $\boxed{\dfrac{dV}{dt} = 0}$

# Riemann Wave Solution

$$\boxed{V(t) = F(\zeta)}$$

$$\boxed{x(t) = \zeta + tF(\zeta)}$$

# Breaking coordinate

$$\boxed{x_{br} = \zeta_{br} + TF(\zeta_{br})}$$

**where** $\zeta_{br}$ **is extreme of** $dF / d\zeta$

**see, breaking time**

$$\boxed{T = \frac{1}{(-dV / dx)_{\max}} = \frac{1}{(-dF / d\zeta)_{\max}}}$$

# Decomposition

$$\zeta = \zeta_{br} + \varepsilon \qquad\qquad \varepsilon \to 0$$

**Taylor series of Riemann wave in the vicinity of breaking**

$$x = \zeta_{br} + \varepsilon + TF(\zeta_{br} + \varepsilon) \approx \zeta_{br} + \varepsilon +$$

$$+T\left[ F(\zeta_{br}) + \varepsilon \underbrace{\boxed{\frac{dF}{d\zeta}(\zeta_{br})}}_{=-1/T} + \frac{\varepsilon^2}{2} \underbrace{\frac{d^2F}{d\zeta^2}(\zeta_{br})}_{=0} + \frac{\varepsilon^3}{6} \frac{d^3F}{d\zeta^3}(\zeta_{br}) + \ldots \right]$$

**Finally**

$$\boxed{x \approx x_{br} + T\left[ \frac{\varepsilon^3}{6} \frac{d^3F}{d\zeta^3}(\zeta_{br}) \right]}$$

**So**

$$\varepsilon = \left[\frac{6}{Td^3F/d\zeta^3}\right]^{1/3}(x - x_{br})^{1/3}$$

**Similar Taylor series for function, V**

$$V(x,T) = F(\zeta_{br} + \varepsilon) \approx F(\zeta_{br}) + \varepsilon\frac{dF}{d\zeta}(\zeta_{br}) + \ldots = F(\zeta_{br}) - \frac{\varepsilon}{T}$$

**In breaking point the wave shape has a singularity**

$$V(x,t) - V_{br} \approx \sqrt[3]{x - x_{br}}$$

**This singularity leads to power spectrum** $k^{-4/3}$

**The same for η due to V(η)**

# Solitons

$$\boxed{\frac{\partial u}{\partial t} + |u|^m \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0}$$

**m > 0 – any integer**

## Soliton

$$\boxed{u \sim \left\{ c \bullet \mathrm{sech}^2 \left[ \frac{\mathrm{m}\sqrt{\mathrm{c}}}{2}(x - ct) \right] \right\}^{1/m}}$$

## Momentum

$$\boxed{E = \int u^2 dx \sim c^{\frac{2}{m} - \frac{1}{2}}}$$

**m < 4 (m > 4)**

**E increases (decreases) with amplitude, c**

**m= 4** **M does not depend from c**

**Solitons in critical and supercritical cases are unstable and blow-up**

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# Numerical study of blow-up and dispersive shocks in solutions to generalized Korteweg–de Vries equations

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## HIGHLIGHTS

- Numerical study of soliton stability in critical and supercritical generalized KdV equations.
- Numerical identification of the blow-up mechanism.
- Numerical study of the small dispersion limit and the $\epsilon$-dependence of the blow-up time.

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## ABSTRACT

We present a detailed numerical study of solutions to general Korteweg–de Vries equations with critical and supercritical nonlinearity, both in the context of dispersive shocks and blow-up. We study the stability of solitons and show that they are unstable against being radiated away and blow-up. In the $L_2$ critical case, the blow-up mechanism by Martel, Merle and Raphaël can be numerically identified. In the limit of small dispersion, it is shown that a dispersive shock always appears before an eventual blow-up. In the latter case, always the first soliton to appear will blow up. It is shown that the same type of blow-up as for the perturbations of the soliton can be observed which indicates that the theory by Martel, Merle and Raphaël is also applicable to initial data with a mass much larger than the soliton mass. We study the scaling of the blow-up time $t^*$ in dependence of the small dispersion parameter $\epsilon$ and find an exponential dependence $t^*(\epsilon)$ and that there is a minimal blow-up time $t_0^*$ greater than the critical time of the corresponding Hopf solution for $\epsilon \to 0$. To study the cases with blow-up in detail, we apply the first dynamic rescaling for generalized Korteweg–de Vries equations. This allows to identify the type of the singularity.

$$\boxed{\frac{\partial u}{\partial t} - u^m \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0}$$

***In our 3-layer flow a sign "minus" if m > 3***

# Conclusion

$$\boxed{\frac{\partial u}{\partial t}+6|u|\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0}$$

- **Solitons of both polarities**
- **N-soliton solutions for the same polarity**
- **Inelastic interaction of solitons of opposite polarities**