Funding:

Topological properties of aftershock clusters in a viscoelastic model of quasi-brittle failure

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Open chatroom with the authors on **Monday, 04 May 2020, 14:00-15:45 CET** at:

https://meetingorganizer.copernicus.org/EGU2020/EGU2020-4948.html

Comments are welcomed befor 30 May 2020.

Author notes:

This is the adaptation of a poster, intended to optimize the opportunities offered by the new display format: halfway between a poster and slide show. Find the original poster at the end of the document.

Officially, the chatroom will be open on **Monday, 04 May 2020, 14:00-15:45 CET** at the following link:

https://meetingorganizer.copernicus.org/EGU2020/EGU2020-4948.html

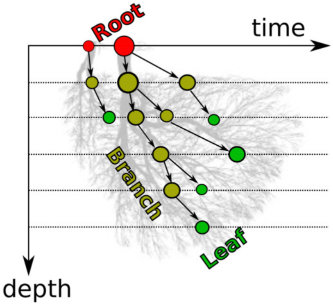
The presenting author will be also available from **10:45-12:30 CET** on the same day **4 May 2020**, at chatroom:

https://meetingorganizer.copernicus.org/EGU2020/EGU2020-6334.html

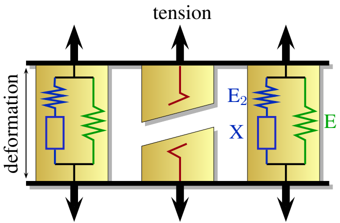
Please, feel free to leave comments at any other time **before 30 May 2020**, at the same links.

Note: blue texts are hyperlinks redirecting to supplementary material in this document and external links.

Material failure at different scales and processes can be modeled as an emergent feature of **micromechanical** systems in terms of **avalanche dynamics**. Among experimental observations, event-event triggering —**aftershocks**— is a common feature in seismological catalogs, acoustic emission experiments [*1*] and even other phenomena. In parallel, the statistical properties of triggering in such catalogs are often modeled as stochastic **epidemic branching** or **linear Hawkes** processes [*4,5*]. In the **micromechanical** approach, **viscoelastic** stress transfer and after-slip are among the proposed mechanism behind triggering and aftershocks.

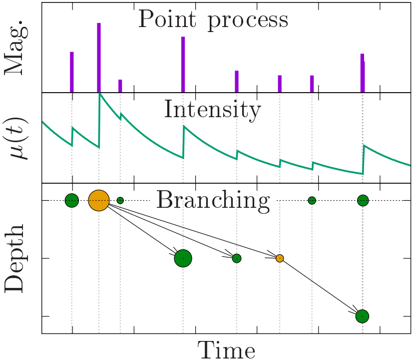


Here we address a simple question:
Do aftershock sequences obtained in micromechanical models agree with the predictions and ideas behind the epidemic branching framework?



We introduce two **fibrous models** as prototypes of **viscoelastic fracture** [2] which (i) provide an analytical explanation to the acceleration of activity in absence of critical failure observed in acoustic emission experiments [*3*]; (ii) reproduce the typical spatio-temporal properties of triggering found in field catalogs, acoustic emission experiments; and (iii) agree with the one-to-one causality established in epidemic models, but display discrepancies with the branching topological properties predicted by stochastic models. These are probably caused by physical constrains and nonstationary parameters.

[1] J. Baró et al., *Phys. Rev. Lett.* 110 (8), 088702 (2013);
[2] J. Baró, J. Davidsen, *Phys. Rev. E* **97** (3), 033002 (2018);
[3] J. Baró, et al., *Phys. Rev. Lett.* **120** (24), 245501 (2018);
[4] J. Baró, *J. of Geophys. Res.: Solid Earth*, **125**, e2019JB018530 (2020);
[5] S. Saichev, et al., *Pure and App. Geoph.*, **162** (6), 1113-1134 (2005).



The Epidemic Aftershock Paradigm [4]:

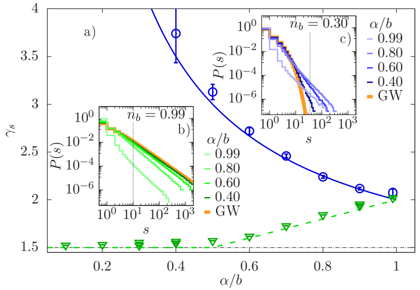
- Catalogs are modeled as a **linear Hawkes** process (ETAS):

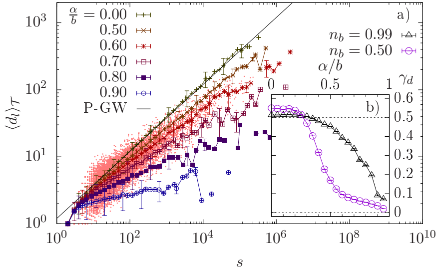
$$\mu(t, \mathbf{r}, m) = \rho(m)\mu_0(t, \mathbf{r}) + \sum_{i|t_i<t} \rho(m) n(m_i) \Psi_i(t, \mathbf{r}|t_i, \mathbf{r_i})$$

considering: - Gutenber-Richter: $\rho(m) \sim 10^{-b(m-m_c)}$
- productivity law: $n(m_i) := k_c 10^{\alpha(m_i-m_c)}$

- ..or specifically as a Galton Watson **branching** process ..

2. Epidemic Aftershock Sequence Models (ETAS) $\longleftrightarrow$ Galton-Watson Branchin





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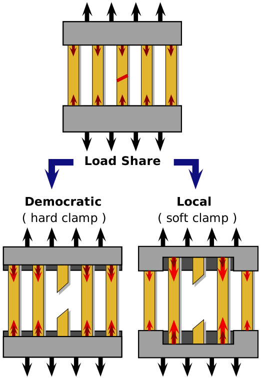
- ..or specifically as a Galton Watson **branching** process exhibiting a transition from Poisson (swarm-like) clusters to scale-free (burst-like) offspring depending on α/b. (details)

$$X_i(m_i) \sim \text{Pois.}(n(m_i)); \text{ as GW :}$$

$$P(X_i = x) = \frac{b}{\alpha} \frac{k_c^{\frac{b}{\alpha}}}{x!} \Gamma\left(x - b(\alpha, k_c)\right)$$

only depends on $0 < \alpha/b \leq 1$ and $n_b \equiv k_0/\left(1 - \frac{\alpha}{b}\right)$.

*Notice that the branching interpretation explicitly considers **one-to-one causal links**. The Hawkes interpretation does not.



Consider a bundle of a number M of equal parallel fibers $\{l\}$: $\varepsilon_l = \int J d\sigma_l$;

each with a random strength S_l;

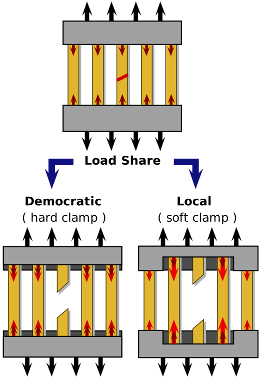
A **global load** σ is shared in individual **local loads**: $\sigma_l(t) = w_l(t)\sigma(t)$;

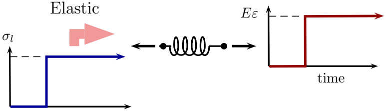
(i) At time t a fiber is broken if $\exists t' | S_l < \sigma_l(t' < t)$; if so: $w_l(t) \to 0$;

(ii) Conservation imposes: $\sum_l w_l = 1 \;\to$ load must be redistributed;

(iii) Sharing Rule: **democratic (DFBM)** (mean field)[2]: $w_l(t) = N^{-1}_{S<\sigma(t)}$

local (LFBM) ($k = 4$ n.n. in a 2d-lattice): $\delta\sigma_{j\in \mathrm{nn}(l)} = \sigma_l/k$;





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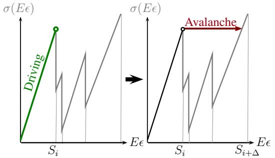
(iv) Creep compliance ($\Delta\epsilon(t) = J(t)\Delta\sigma_l$): **elastic** kernel : $J_{\mathrm{EL}}(t) = 1/E$;

ELASTIC DFBM avalanche:



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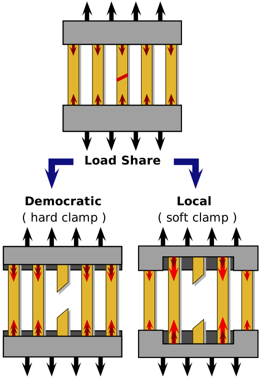
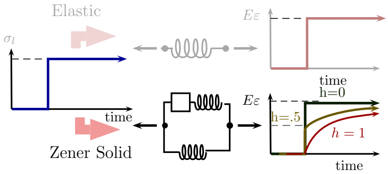
- Avalanches are caused by simultaneous failure of several fibers (with similar strength in the example of DFBM).





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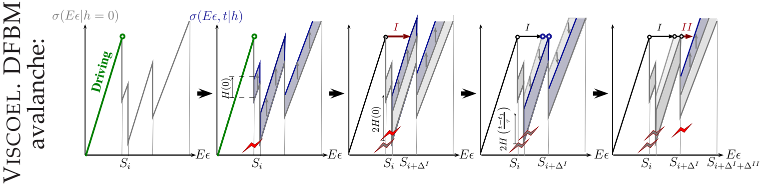
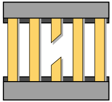
(iv) Creep compliance: **viscoelastic**: $J_{GZ}(t; h, \alpha) = \frac{1}{E}\left(1 - h\mathbf{E}_\alpha\left(\frac{t}{\tau}\right)\right)$;

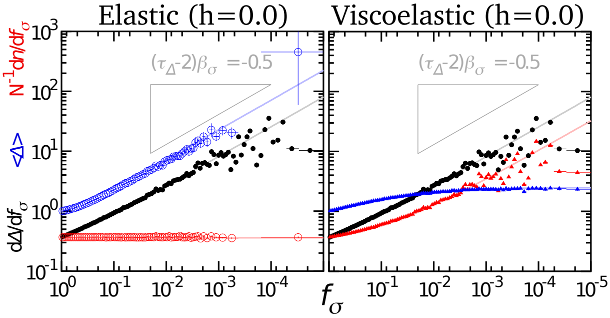


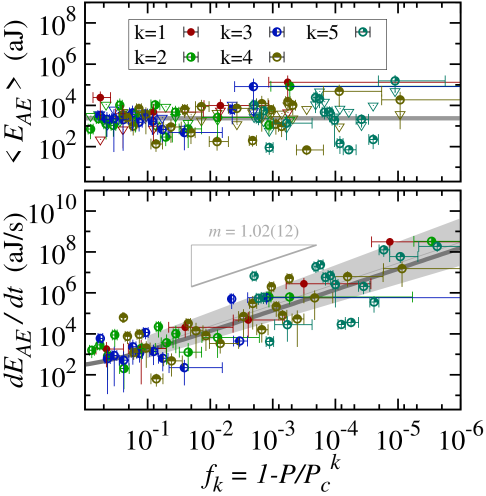
- Avalanches are split in causally connected clusters.
- The magnitude of **transient hardening** (h) controls avalanche statistics.

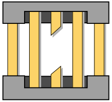
Democratic Fiber Bundle Model



- Acceleration to failure ($d\Delta/df_\sigma(f_\sigma)$) is invariant [2], but
- viscoelasticity trades **critical avals.** for **foreshocks** [2]. ↓
- Observed in acoustic emission experiments [3]. →

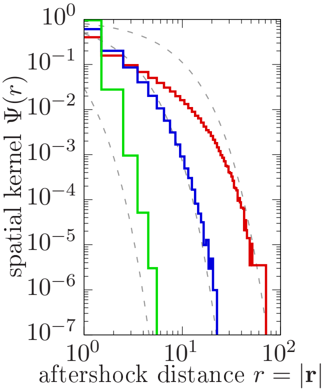


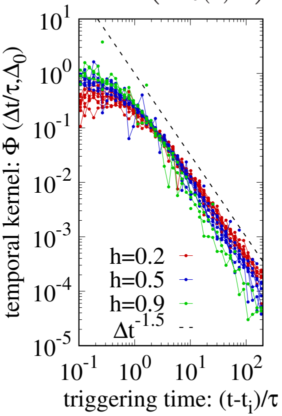


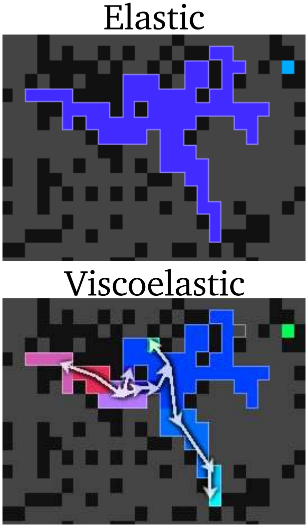


2-D Local Fiber Bundle Model

- Kernel: $\Psi_i(t, \mathbf{r}|t_i, \mathbf{r}t_i) \approx \dfrac{d\mathbf{E}_\alpha\left((t-t_i)/\tau\right)}{dt} \exp\left(\dfrac{-|\mathbf{r}-\mathbf{r_i}|}{r_0(h)}\right).$

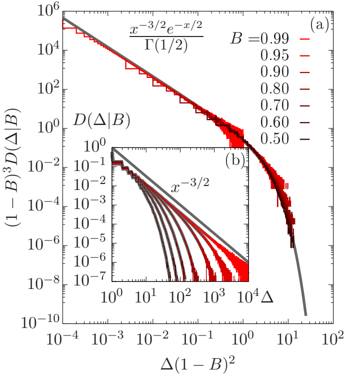






DEMOCRATIC FIBER BUNDLE MODEL

- All avalanche **sizes** are i.i.d. as **Borel** depending on f_σ: $\rho(\Delta; f_c) = (f_c\Delta)^{\Delta-1}e^{-f_c\Delta}/\Delta!$ (details).



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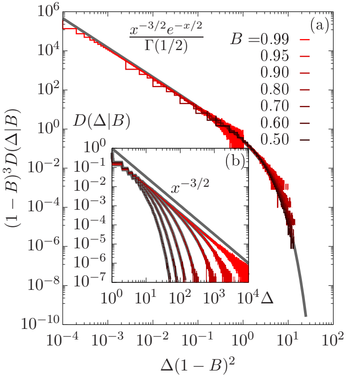
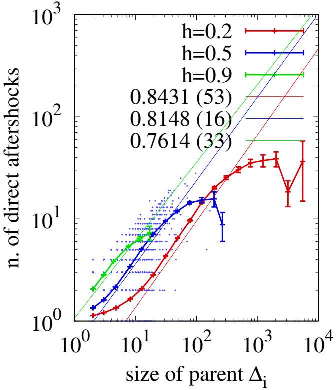
- Considering $m = \log_{10}\Delta$ this implies b-value $= 0.5$; $\alpha \approx 0.8$ ($\alpha/b = 1.6 > 1$).





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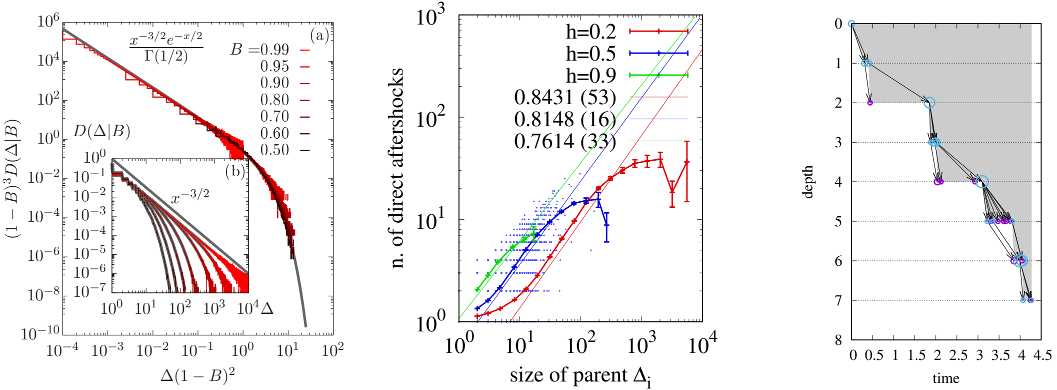
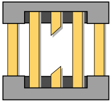
- Considering $m = \log_{10}\Delta$ this implies b-value $= 0.5$; $\alpha \approx 0.8$ ($\alpha/b = 1.6 > 1$).
- Triggering is sorted by parent (see model) rendering unrealistically (non-ETAS) 'combed' trees;

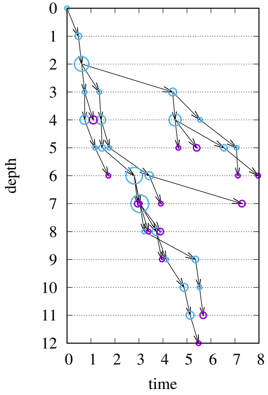


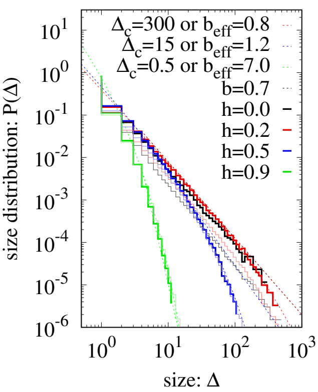


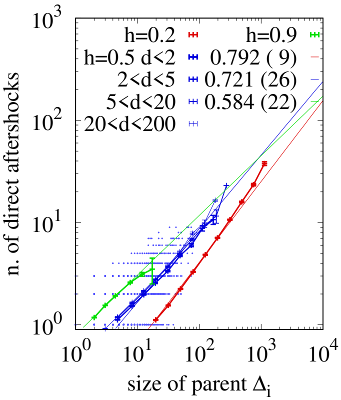
2-D Local Fiber Bundle Model

- No analytic solution. Critical failure for $S_i \sim \text{Weibull}(k(h))$, being $k(0) = 1.36$.

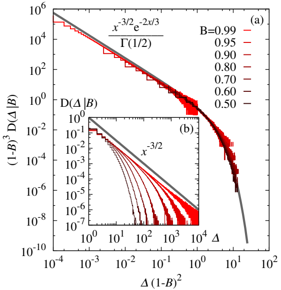
  $P(\Delta) \sim \Delta^{-1-b} \exp(-(\Delta/\Delta_c)^b)$ with $b \sim 0.7$, a cutoff $\Delta_c(h)$ & productivity exponent $\alpha(h)$.

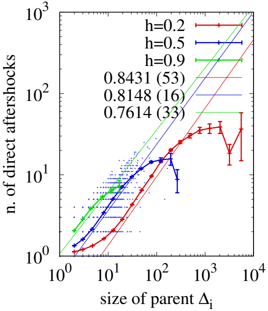


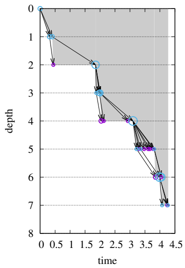


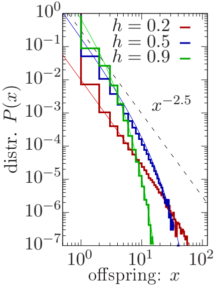


6. Triggering in the FBM with **democratic** load sharing (DFBM)









The exponent values in the DFBM imply: $\alpha/b = 1.6 > 1$!! (unphysical)

BUT, failure is sub-critical

(no divergence of $\langle$number of offsprings$^2\rangle$)

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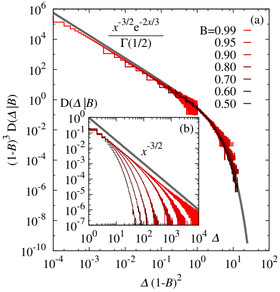

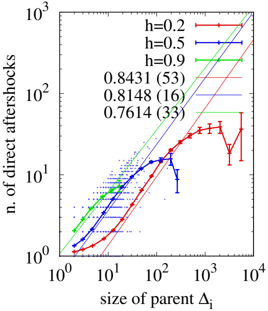

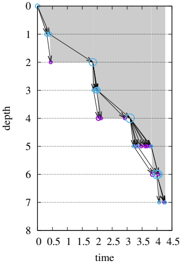

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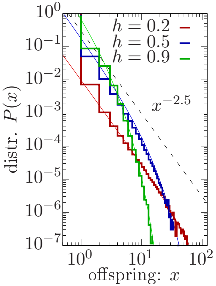
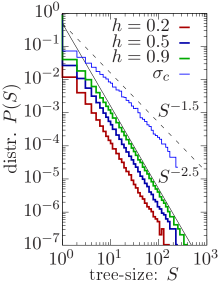
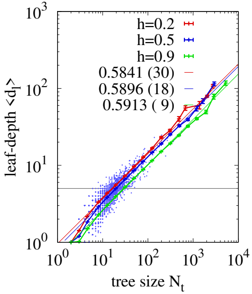
BUT, failure is sub-critical
(no divergence of $\langle$number of offsprings$^2\rangle$)

Distr. of tree-size (S) $\sim$ Distr. of avalanche sizes (Δ)
(imposed by physical constrains [2])

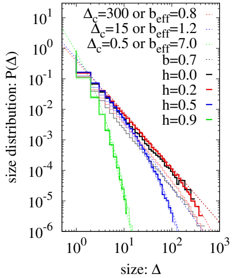
Only swarm-like clusters ($d_l \sim S^{0.5}$)

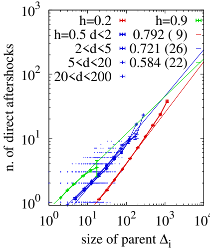


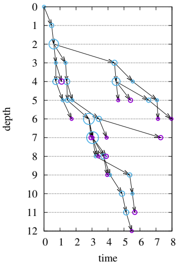


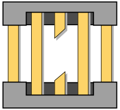


6. Triggering in the FBM with **local** load sharing (LFBM)







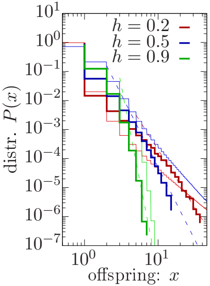


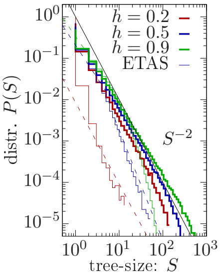
Power-law avalanches can occur with $h > 0$ (unlike the DFBM)

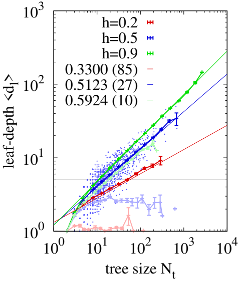
Transition from swarms to bursts is observed by tunning h.

Global exponents (table here) do not fully satisfy expectations from ETAS.

This is corrected by considering the non-stationary evolution of statistical parameters corresponding to the phase transition. (WIP not shown here)





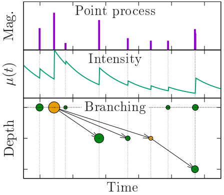


CRM CENTRE DE RECERCA MATEMÀTICA

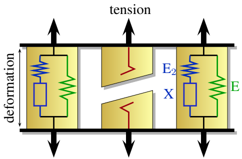
- **Epidemic models** (ETAS) interpret aftershock sequences as:

  (1) linear Hawkes' processes with an intensity factorized in empirical terms accounting for: mainshock mag., and spatio-temporal and mag. distribution.

  (2) a Galton-Watson **branching forests** with one-to-one causal links, independent offspring production and a transition between **Poisson** and **power-law** offspring distribution.



- Avalanches in **micromechanical models** of viscoelastic fracture can be triggered by external driving (mainshocks in terms of seismology) or one-to-one causal triggering (aftershocks) and reproduce *qualitatively* the expectations of the ETAS model.



  (1) Aftershocks are defined by one-to-one causal links, leading to branching.

  (2) Explain the origin of **acceleration** in absence of **critical** avals. at failure

  (3) Reproduce the expected **cluster** classification depending on physical parameters. We find **bursts** for high viscosity (h) and **swarms** for low.

  (4) To reproduce *quantitatively* all topological features we must consider physical constrains and non-stationary parameters in the ETAS model (WIP).

Thanks: AXA, AXA Research Fund, Through Research, Protection, NSERC CRSNG

Supplementary Information

Epidemic Type Aftershock Sequence Model (ETAS) [Y. Ogata, JASA (1988)] :

$$\Psi(\Delta, t, \mathbf{r}|m_i, t_i, \mathbf{r_i}) = \rho(\Delta) n_b(\Delta_0)\, \Psi_t(t - t_0) \Psi_r(||\mathbf{r} - \mathbf{r_0}||)$$

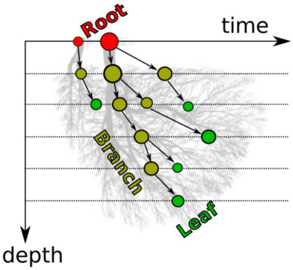
$$\left.\begin{aligned} \rho(\Delta) &\sim \Delta^{-1-b} \\ n_b(\Delta_0) &\sim \Delta_0^{\alpha} \end{aligned}\right\}$$

Results depend only on $\langle n_b \rangle$ and α/b.

$0 \leq \alpha/b < 0.5$ *typical* Galton-Watson

$0.5 \leq \alpha/b < 1$ special case

[A. Saichev, et al. (2015)]



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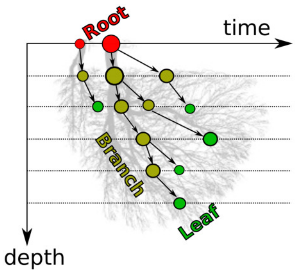

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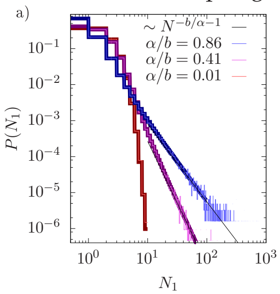
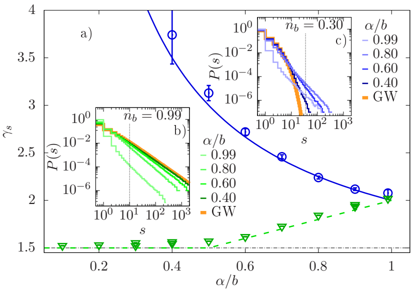
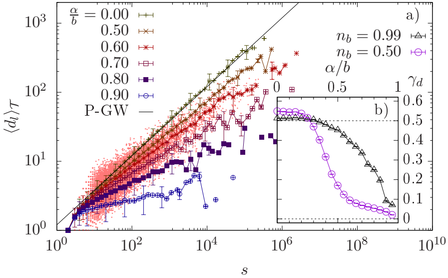
Distr. of offspring



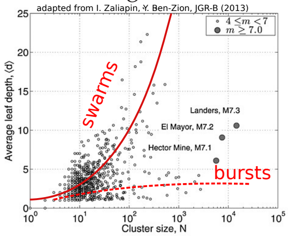
Distr. of tree-sizes

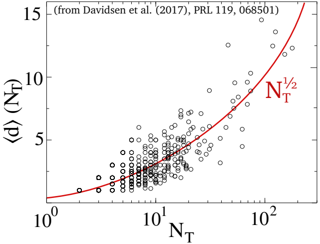


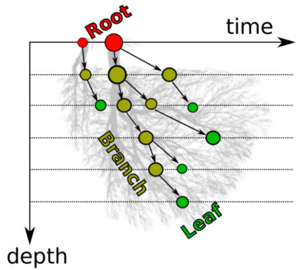
Relation: sizes vs. leaf-depths



Field catalog class.: **Bursts** ($d_l \lesssim 5$) and **Swarms** ($d_l \sim S^{\gamma_d}$)







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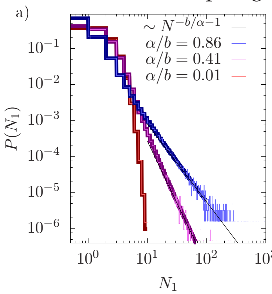
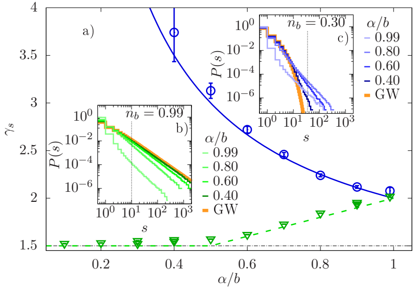
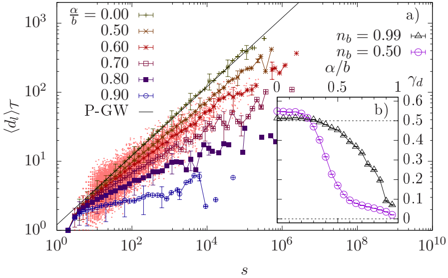
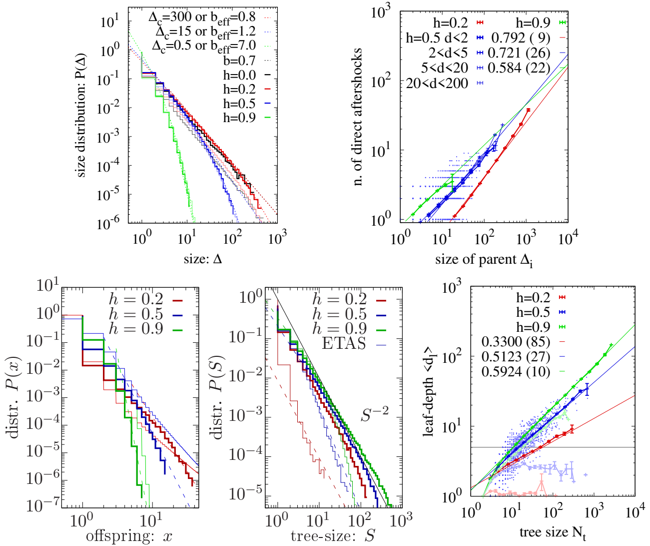






Exponent table for LFBM and ETAS expectations

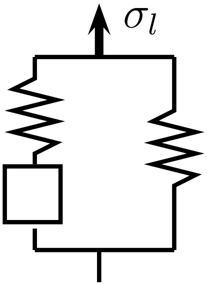


h	0.2	0.5	0.9
b_{eff}	0.8	1.2	7.0
α	0.78	0.72	0.58
$\dfrac{\alpha}{b_{\text{eff}}}$	0.975	0.600	0.082
n_b	0.171	0.413	0.675
$\gamma_x^{\text{mod}} = 1 + b/\alpha$	2.03	2.67	13.1
$\hat{\gamma}_x$	4	5	13
$\gamma_s^{\text{mod,h}} = 1 + b/\alpha$	2.03	2.67	13.1
$\gamma_s^{\text{mod,l}} = 1 + \alpha/b$	1.96	1.60	1.08
$s_c^{\text{mod}} = (1 - n_b)^{1/(\alpha/b-1)}$	1775	3.78	3.40
$\hat{\gamma}_s$	2.1	1.9	1.8
γ_d^{sim}	0	0	0.5
$\hat{\gamma}_d$	0.33	0.51	0.59

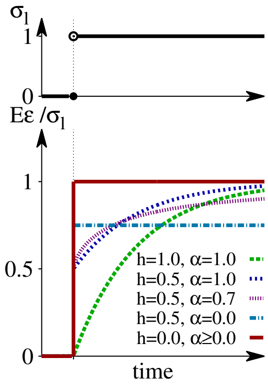
Adds **dissipation**, **temporal scales** and **power-law memory**.
Reproduces the response of certain bulk amorphous materials

$$\sigma_m = E_m \varepsilon_m$$

$$\sigma_X = X \frac{d^\alpha}{dt^\alpha} \varepsilon_X$$



$$\sigma_e = E\varepsilon$$



Constitutive equation for the generalized Zener Element:

$$\left[1 + \frac{X}{E_m} \frac{d^\alpha}{dt^\alpha}\right] \sigma_l = \left[1 + \frac{X(E_m + E)}{E_m E} \frac{d^\alpha}{dt^\alpha}\right] E\varepsilon.$$

$$\mathbf{H}_\alpha(t/\tau) := \frac{E_m}{E_m + E} \mathbf{E}_\alpha \left(- \left(\frac{t}{\tau} \right)^\alpha \right) \begin{cases} \mathbf{H}_\alpha(0) & = & \dfrac{E_m}{E_m + E} := h \\ \mathbf{H}_\alpha(t \gg \tau) & \to & 0 \end{cases}$$

Creep compliance:

$$J_{GZ}(t) = \frac{1}{E} \left(1 - \mathbf{H}_\alpha \left(\frac{t}{\tau} \right) \right)$$

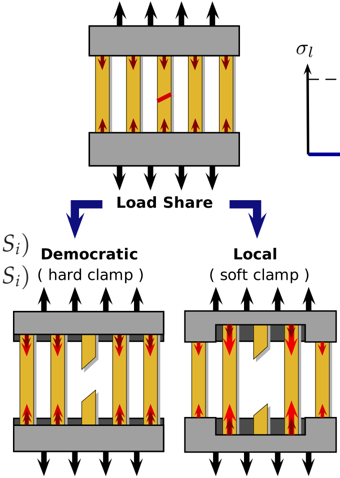
[J. Baró & J. Davidsen, PRE (2018)]

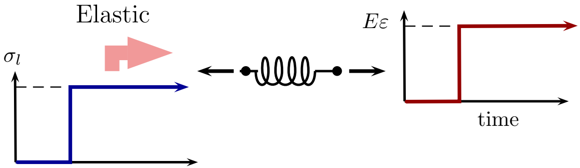
Fiber Bundle Models (FBM):

Interaction (*sharing rule*):

Fiber Deformation (*overdamped*):

Brittle fibers of stochastic strengths S_i:





Local load at each fiber:

$$\sigma_l = \begin{cases} w_l(t)\sigma & (E\varepsilon < S_i) \\ 0 & (E\varepsilon \geq S_i) \end{cases}$$

AVALANCHE: collective failure of Δ fibers

Micromechanical model with triggering: Viscoelastic fracture

[*J. Baró & J. Davidsen, PRE (2018)*]

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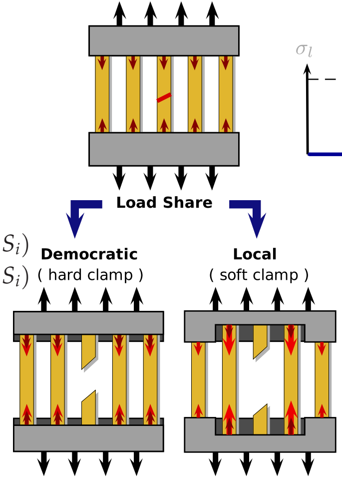

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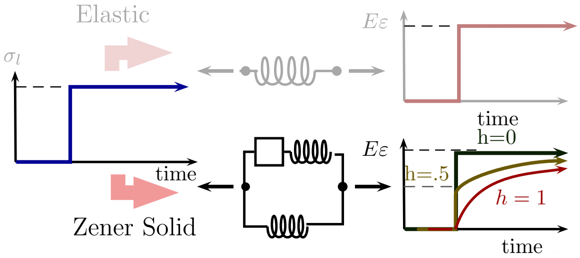
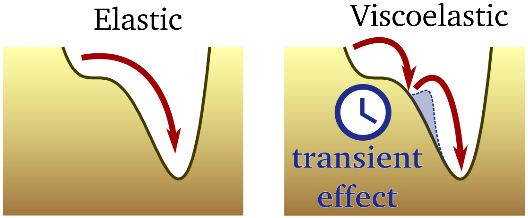
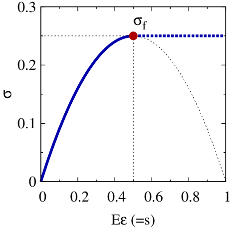


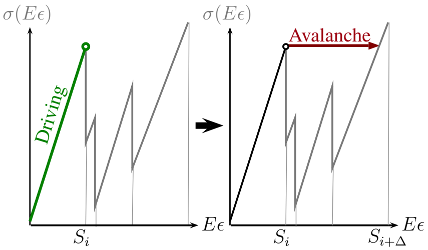


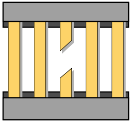
ELASTIC ($E\varepsilon = \sigma_l$)

Constitutive equation:

$$\boxed{\sigma(E\varepsilon) = \left(\frac{N_0}{N(S_i > E\varepsilon)}\right)^{-1} E\varepsilon}$$







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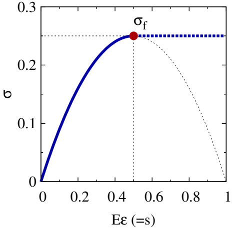
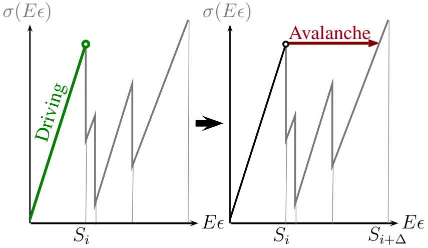
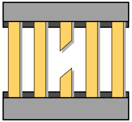


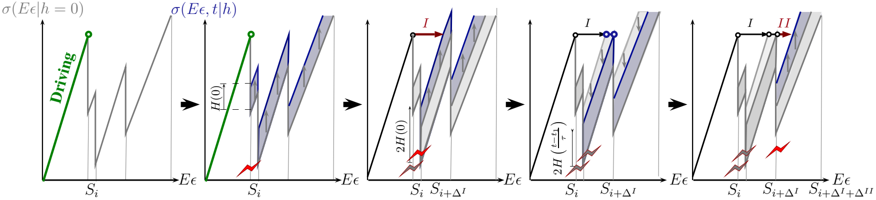




VISCOELASTIC ($E\varepsilon = (1 - f(t))\sigma_l$)

Constitutive equation:

$$\boxed{\sigma(E\varepsilon, t) = \left(\frac{N_0}{N(S_i > E\varepsilon)} - \sum_{S_j < E\varepsilon} \phi_j\left(t - t_j\right)\right)^{-1} E\varepsilon}$$



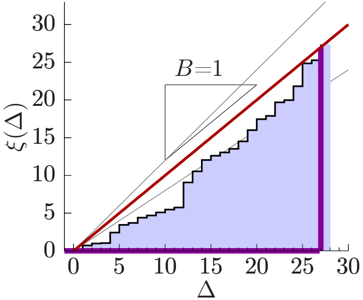
CENTRE DE RECERCA MATEMÀTICA

From constitutive equation: An avalanche start at S_i, stops when: $\sigma(\varepsilon) \geq \sigma(S_i)$

Universal Avalanche Definition:

$$\xi(\Delta_i) > B(S_i|h)\Delta_i$$

(hitting times)



$\xi(\Delta)$: **Poisson counting process of Δ steps.**

[*Baró & Davidsen, PRE (2018)*]

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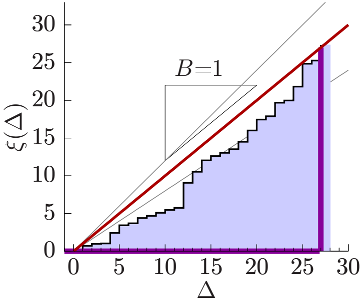

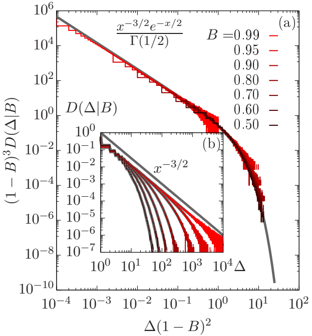

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$B = 1$: **Critical** $D(\Delta; B) \sim \Delta^{-1.5}$.

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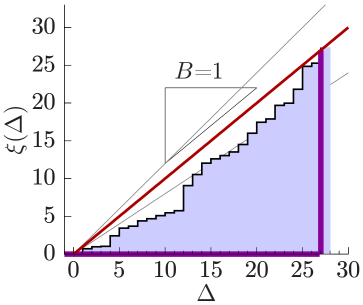

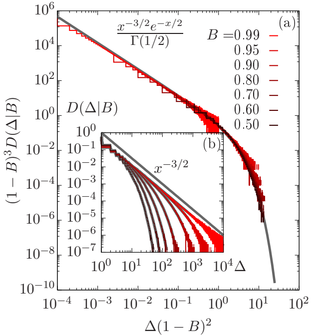

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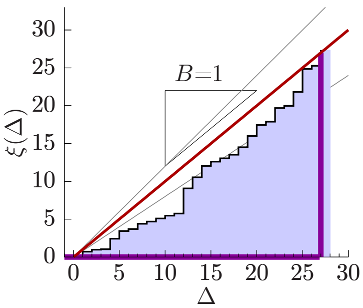

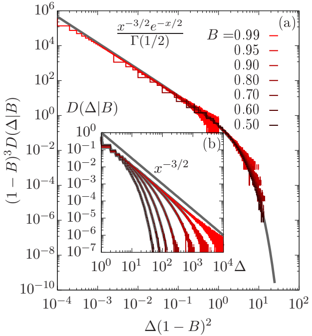

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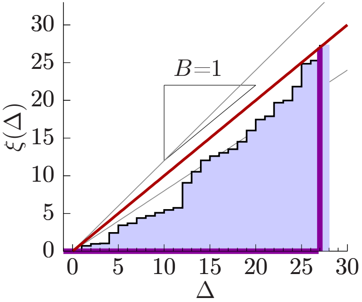
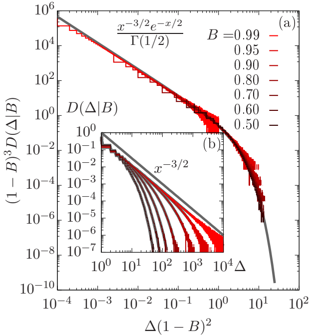
Critical failure for $h = 0$.

Subcrit. failure for $h > 0$.

[*Baró & Davidsen, PRE (2018)*]





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Topological properties of aftershock clusters in a viscoelastic model of quasi-brittle failure

Jordi Baró [a,b], Jörn Davidsen [a], Alvaro Corral [b]

[a] Complexity Science Group, Dept. of Physics and Astronomy, Univ. of Calgary, Calgary, AB, T2N 1N4, Canada.

[b] Centre for Mathematical Research (CRM), Barcelona, 08193, Spain.

1) Viscoelasticity & Aftershocks

Material failure at different scales and processes can be modeled as an emergent feature in terms of **avalanche dynamics** in **micromechanical** systems. Event-event triggering —**aftershocks**— is common in seismological catalogs and acoustic emission experiments [1] among other phenomena. Stochastic **branching** and **linear Hawkes** processes are used to model the statistical properties of catalogs. In the **micromechanical** approach, **viscoelastic** stress transfer and after-slip are among the proposed mechanism of aftershocks.

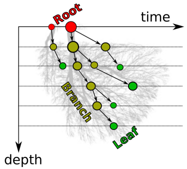
Here we ask this simple question:

Do aftershock sequences in micromechanical models agree with such epidemic branching paradigm?

We introduce two **fibrous models** as prototypes of **viscoelastic fracture** [2] which (i) provides an analytical explanation to the acceleration of activity in absence of critical failure observed in acoustic emission experiments [3]; (ii) reproduce the typical spatio-temporal properties of triggering found in field catalogs, acoustic emission experiments; but (iii) display discrepancies with the branching topological properties predicted by stochastic models, probably due to physical constrains.

[1] J. Baró et al., *Phys. Rev. Lett.* 110 (8), 088702 (2013).
[2] J. Baró, J. Davidsen, *Phys. Rev. E* **97** (3), 033002 (2018).
[3] J. Baró, et al., *Phys. Rev. Lett.* **120** (24), 245501 (2018).
[4] J. Baró, *J. of Geophys. Res.: Solid Earth*, **125**, e2019JB018530 (2020);
S. Saichev, et al., *Pure and App. Geoph.*, **162** (6), 1113-1134 (2005).

2) Epidemic Aftershock Sequence Models (ETAS) ⟷ Galton-Watson Branching (GW) [4]



- Catalogs are modeled as a **linear Hawkes** process (ETAS):

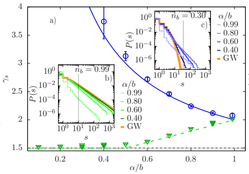
$$\mu(t,\mathbf{r},m) = \rho(m)\mu_0(t,\mathbf{r}) + \sum_{i|t_i<t} \rho(m)n(m_i)\Psi_i(t,\mathbf{r}|t_i,\mathbf{r_i})$$

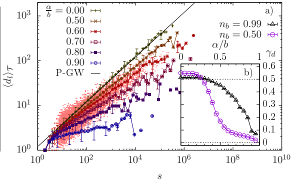
GR: $\rho(m) \sim 10^{-b(m-m_c)}$ & prod. law: $n(m_i) := k_c 10^{\alpha(m_i-m_c)}$

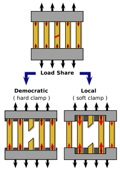
- ..or as GW **branching** process: (Poisson → scale-free offspring)

$$X_i(m_i) \sim \text{Pois.}(n(m_i));\ \text{as GW}: P(X_i = x) = \frac{b}{\alpha}\frac{k_c^{\frac{b}{\alpha}}}{x!}\Gamma\left(x - b(\alpha, k_c)\right)$$

only depends on $0 < \alpha/b \leq 1$ *and* $n_b \equiv k_0/\left(1-\frac{\alpha}{b}\right)$.





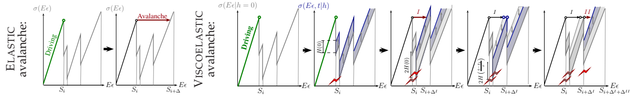


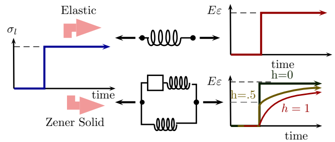
Bundle of M equal fibers $\{l\}$: $\varepsilon_l = \int J d\sigma_l$; local load to each fiber: $\sigma_l(t) = w_l(t)\sigma(t)$; random strength S_l;

(i) Fracture: $w_l(t) \to 0$ if $\exists t' | S_l < \sigma_l(t' < t)$; **(ii)** Conservation: $\sum_l w_l = 1$;

(iii) Sharing Rule: **democratic** (mean-field): $w_l(t) = N^{-1}_{S<\sigma(t)}$ vs. **local** (k nn in 2d-lattice): $\delta\sigma_{j\in \text{nn}(l)} = \sigma_l/k$;

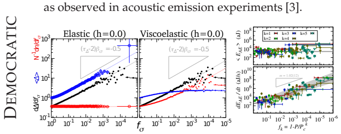
(iv) Mechanical response: **elastic** $J_{\text{EL}}(t) = 1/E$, vs. **viscoelastic**: $J_{\text{GZ}}(t; h, \alpha) = \frac{1}{E}\left(1 - h\mathbf{E}_\alpha\left(\frac{t}{\tau}\right)\right)$;



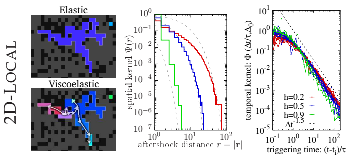


4) Activity and Aftershocks

- Viscoelasticity trades **critical avals.** for **foreshocks** [2]. as observed in acoustic emission experiments [3].

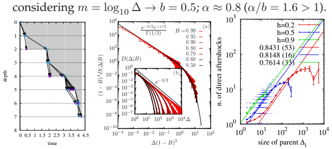


- Aftershock kernel: $\Psi_i(t,\mathbf{r}) \approx \frac{d}{dt}\mathbf{E}_\alpha\left(\frac{t}{\tau}\right)\exp\left(\frac{-|\mathbf{r}|}{r_0(h)}\right)$.

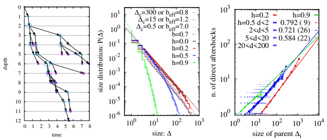


5) Branching Parameters (α/b, n_b)

- 'combed' trees; Borel distr.: $P(\Delta; f_c) = (f_c\Delta)^{\Delta-1}e^{-f_c\Delta}/\Delta!$. considering $m = \log_{10}\Delta \to b = 0.5$; $\alpha \approx 0.8$ ($\alpha/b = 1.6 > 1$).

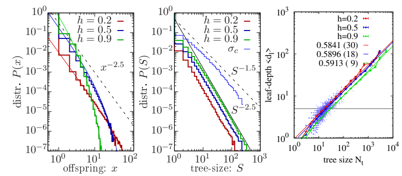


- no analytic solution: crit. fail. at $S_i \sim$ Weibull($\alpha = 1.36$). Size distribution: $\Delta \sim \Delta^{-1-b}\exp(-(\Delta/\Delta_c)^b)$ w. $b \sim 0.7$; cutoff Δ_c & productivity (α) depend on h.



Comparison of trees in DFBM with ETAS model:

- Offspring (X) and tree-size (S) distributions: power law w. $\gamma_s = \gamma_x = 2.5$ ($\alpha/b = 2/3$ in ETAS) From aval. size (Δ)?: $\int P(\Delta; f_c)df_c \Rightarrow S \sim \Delta(h=0)$. ($\gamma = 1.5$ at crit. point)
- Leaf-depths (d_l) scale $\sim S^{0.5}$ like **swarms** (low α/b). Not as **bursts** (d indep. of S) expected for high α/b.
- ETAS needs to account for cutoff (correcting $\alpha/b > 1$) and temporal variations of parameters.



h	0.2	0.5	0.9
b_{eff}	0.8	1.2	7.0
α	0.78	0.72	0.58
$\frac{\alpha}{b_{\text{eff}}}$	0.975	0.600	0.082
n_b	0.171	0.413	0.675
$\gamma_x^{\text{mod}}(\gamma_s^{\text{mod,h}})$	2.03	2.67	13.1
$\hat{\gamma}_x$	4	5	13
$\gamma_s^{\text{mod,l}}$	1.96	1.60	1.08
S_c^{mod}	1775	3.78	3.40
$\hat{\gamma}_s$	2.1	1.9	1.8
γ_d^{sim}	0	0	0.5
$\hat{\gamma}_d$	0.33	0.51	0.59

- In LFBM, triggering trees display a transition from **bursts** to **swarms** (as expected).
- Distributions (and exponent values) disagree with an ETAS model with the fitted parameters.