On the criterion of self-consistency and its feasibility in models of elastic properties of fractured media

V.Yu. Zaitsev, A.V. Radostin

Institute of Applied Physics, Russian Academy of Sciences,
Nizhny Novgorod, Russia

Motivation

- Observation of macroscopically measured elastic properties of rocks are widely used to extract information about the rock microstructure by comparing experimental and predicted elastic moduli

- To perform such comparison, the model should be able to adequately describe the experiments

- A natural criterion of adequacy/correctness of the model is that the output of this model correctly reproduces the measured data if the input of the model also agrees with the experiment

- Although this criterion looks natural, we demonstrate that quite often it is not satisfied by widely used and generally accepted model equations

- In this report we demonstrate examples of pressure dependences for P- and S-wave velocities, for which equations based on the generally accepted model of penny-shape cracks demonstrate drastic inconsistency between the input and output pressure dependences

- The reason of this inconsistency is elucidated and an alternative way of interpretation is proposed

Estimation of crack concentration by comparing dependences of P and S-wave velocities on hydrostatic pressure with model equations

Widely used equations are [Zimmerman, R. W. (1990). *Compressibility of sandstones* (Vol. 29). Elsevier]:

$$V_p^2 = (K + 4G/3)/\rho \qquad V_s^2 = G/\rho$$

K and *G* are the bulk and shear elastic moduli.

For a solid with isotropically oriented dry penny-shape cracks one can obtain the dependences of the elastic moduli on the crack's concentration $\Gamma = N < a^3 >$ (*N* is the number of cracks per unit volume, a is the radius of cracks and <…> denotes averaging over all cracks)

$$\boxed{\frac{K}{K^{hp}} = \frac{(1-2\nu^{hp})\exp(-16\Gamma/9)}{1-2\nu^{hp}\exp(-8\Gamma/5)}} \qquad \boxed{\frac{\nu}{\nu^{hp}} = \exp(-8\Gamma/5)} \qquad \boxed{\Gamma(p) = \Gamma^i \exp(-p/p_c)}$$

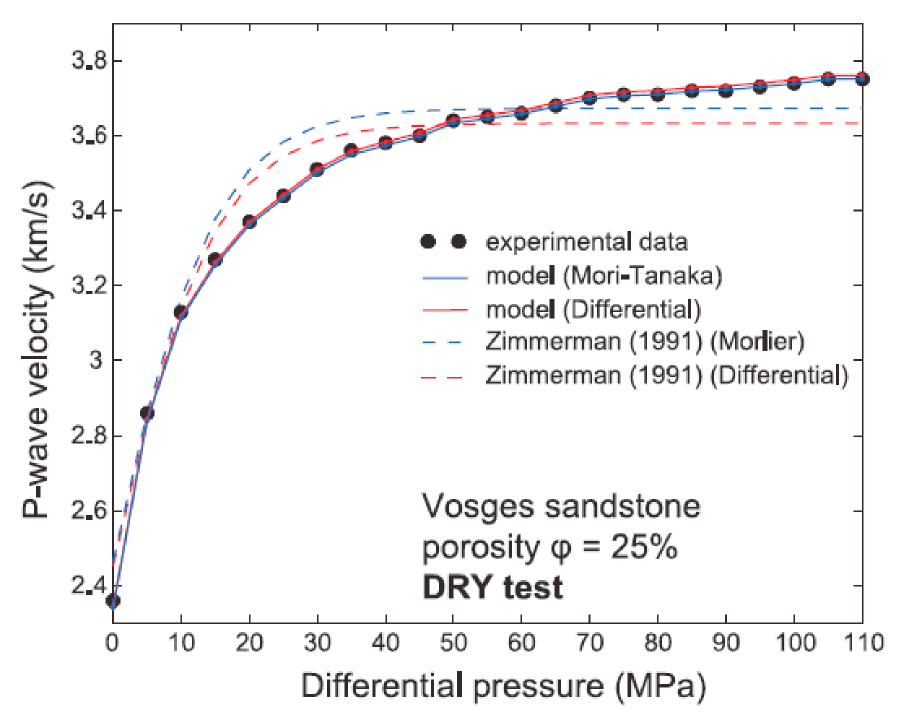
p and p_c are the current pressure and its characteristic value defining pressure-sensitivity of cracks's concentration, indexes i and hp the corresponding quantities at small and maximal pressures, ρ is density.

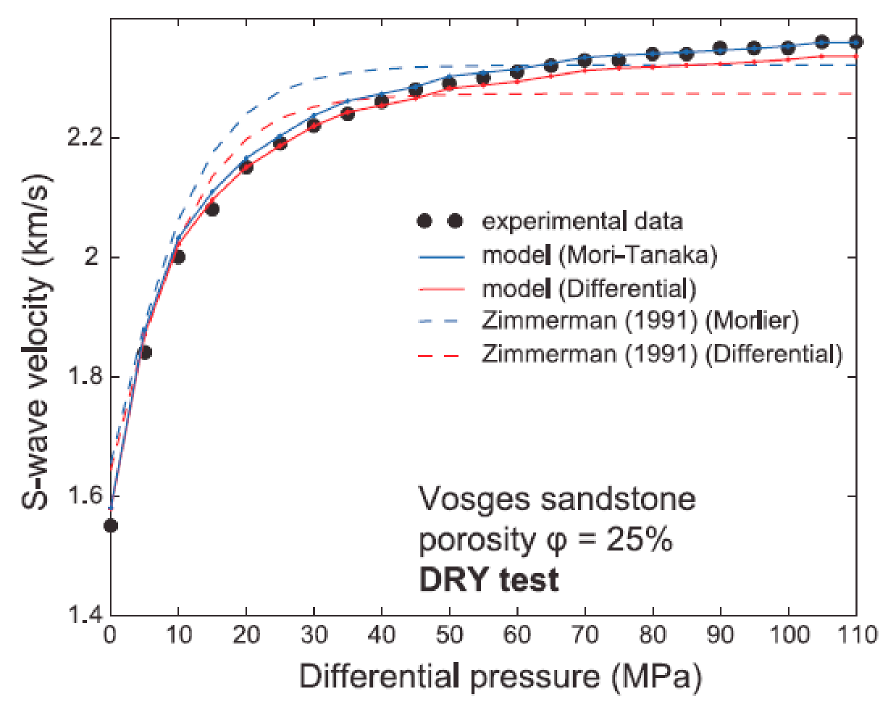
$$\nu = (3K - 2G)/(6K + 2G)$$ is the Poisson ratio.

The values of Γ^i u p_c are determined from comparison of experimental data and given expressions

Experimental data for Vosges sandstone

David E., Zimmerman R. // J. Geophys. Res. 2012. **117**. P. B07210.



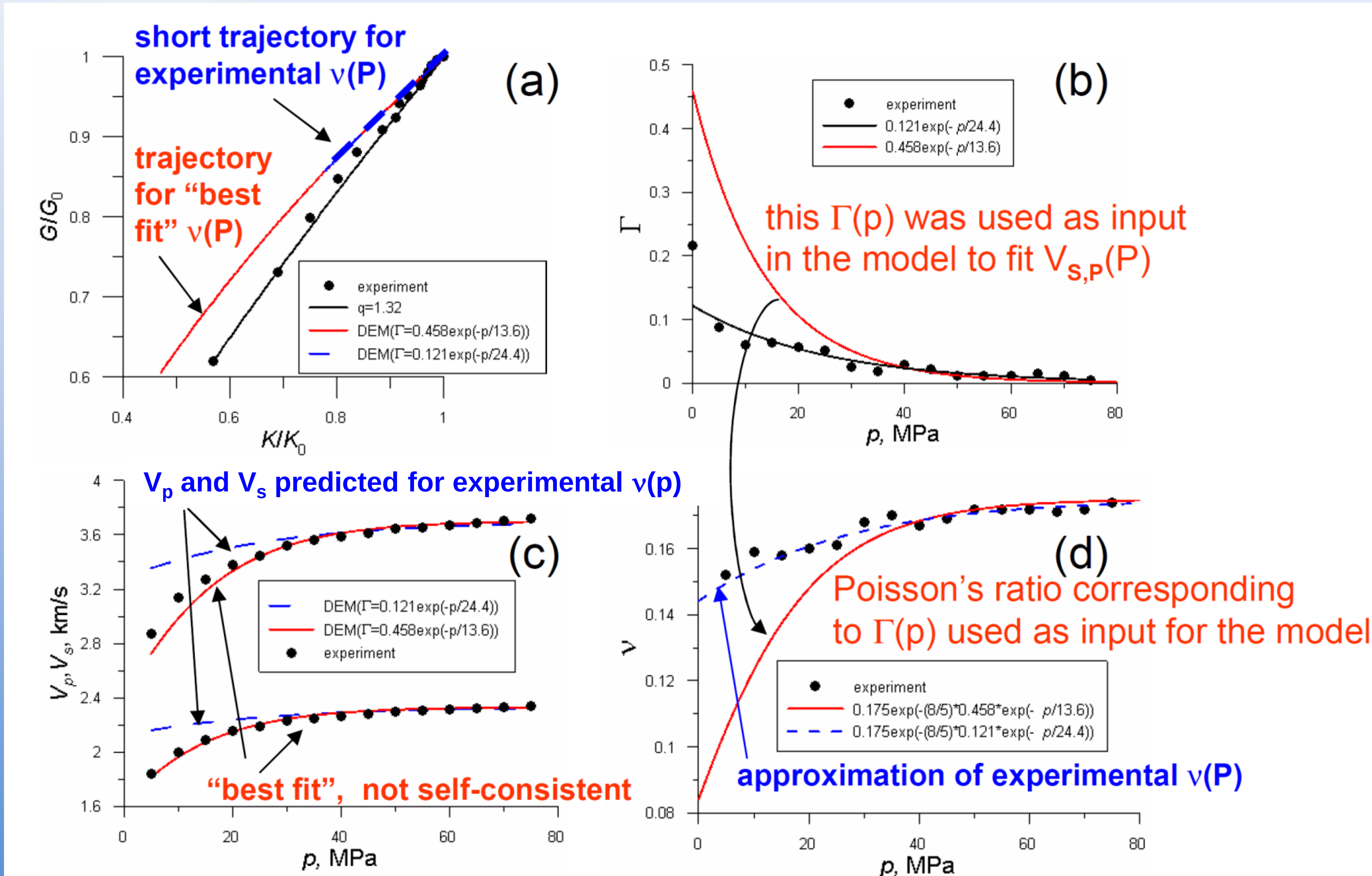


$$\boxed{\Gamma(p) = 0.458 \exp(-p/13.6)}$$

Red solid lines

This Γ(p) apparently gives to the best fit for the pressure dependences!

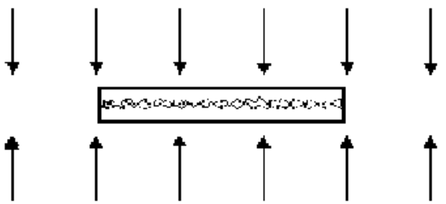
Closer examination reveals violation of self consistency between the Γ(p) and the other moduli assumed at the input of the model and the predicted values



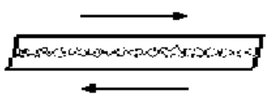
Description of the medium based on the crack model as a soft planar object with decoupled compliance parameters with respect to normal and shear loading

(Zaitsev, Sas, Acustica-Acta Acustica 2000; MacBeth, Geophysics 2004)

$$E_d \equiv \varsigma E \qquad \varsigma << 1$$



$$G_d \equiv \xi G \qquad \xi << 1$$



the Poisson ratio

$$\nu(K,G) = \frac{(3K-2G)}{(6K+2G)}$$

$$\tilde{K} = \frac{K_{eff.}}{K_0} = \frac{1}{1+\frac{1}{3}N_n/(1-2\nu_0)}$$

$$\tilde{G} = \frac{G_{eff.}}{G_0} = \frac{1}{1+\frac{2}{15}N_n/(1+\nu_0)+\frac{2}{5}N_s}$$

$$\tilde{E} = \frac{E_{eff.}}{E_0} = \frac{1}{1+\frac{1}{5}N_n+\frac{4}{15}(1+\nu_0)N_s}$$

K_0, G_0, E_0, ν_0 are matrix parameters (or measured quantities at high values of applied pressure)

$$\left.\begin{aligned} N_n &= \int f(\varsigma)\varsigma^{-1}d\varsigma \\ N_s &= \int f(\xi)\xi^{-1}d\xi \end{aligned}\right\}$$

These parameters are proportional to the values of compressibility to compression and shear, and are close in magnitude to the effective volume of cracks

The different dependence of different modules on N_n and N_s makes it possible to determine **the key property of cracks** – the ratio of normal and shear compliances: ***$q = N_n / N_s$***

In the case of penny-shaped cracks (using $B = Z_n/Z_T$ Kachanov, 1992):

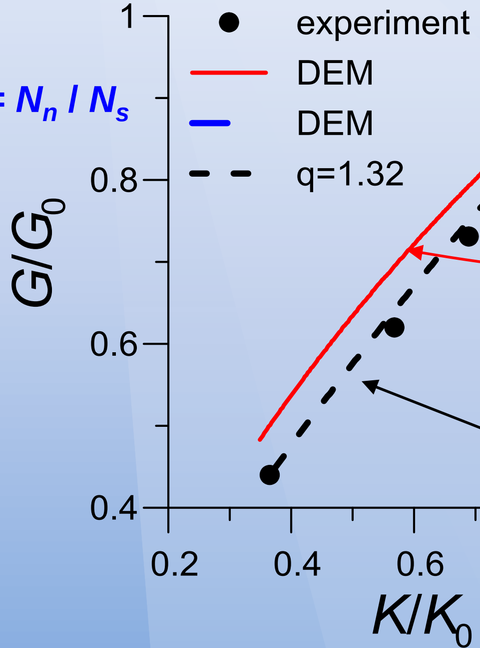
$$\boxed{q = (1+\nu_0)(2-\nu_0) = 2(1+\nu_0)B}$$

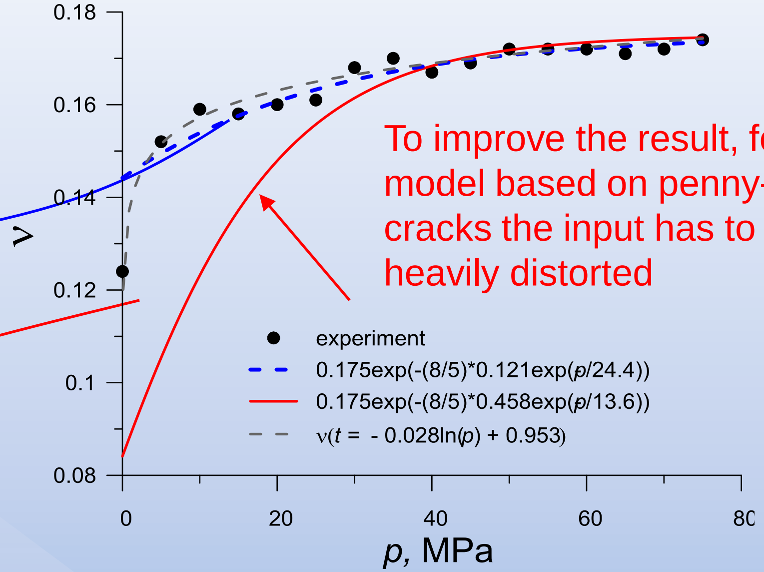
$$\boxed{2 \le q \le 2.25} \quad \text{for} \quad 0 \le \nu_0 \le 0.5$$

Representation of the same data and approximations on the plane (*K*,*G*) – there is no masking influence of the assumed shape of Γ(*p*)

❑ The curves (red and blue dashed) correspond to the expressions for the penny-shape crack model deviate significantly from the experimental points:

q= N $q = N_n / N_s$





To improve the result, for the model based on penny-shape cracks the input has to be heavily distorted

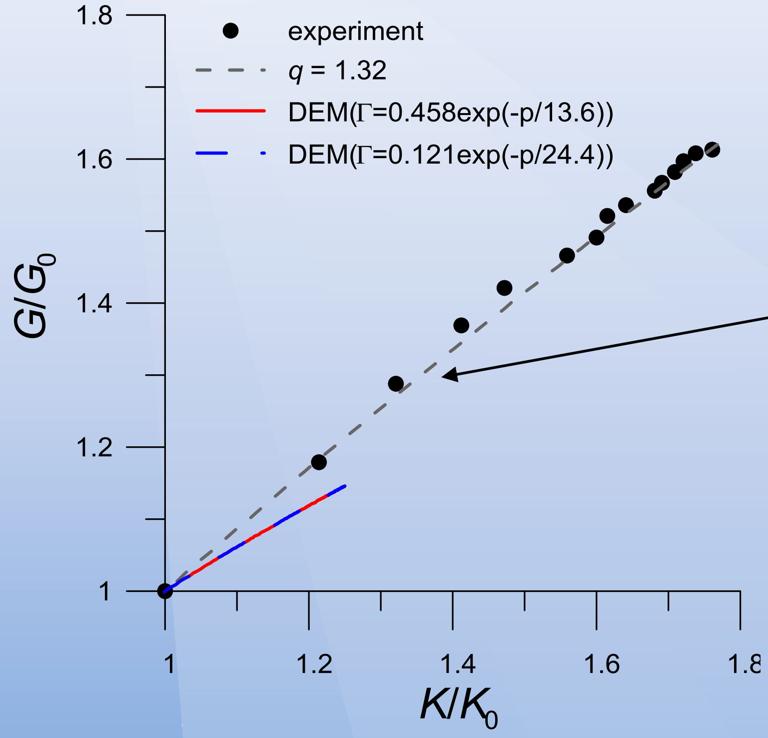
❑ Alternative description based on a model of a medium with crack-like defects characterized by independently introduced normal and shear compliance [1-4] (black line on the diff. models [3])

❑In this case, no auxiliary assumptions about the dependence on pressure are required.

1. *Zaitsev V., Sas P.* // Acustica-Acta Acustica, v.86, 216-228 (2000),
2. *Zaitsev V., Sas P.* // Phys. Mesomech. **7,** 37. (2004)
3. *Zaitsev V., Radostin A., Pasternak E., Dyskin A.* // Int. J. Rock Mech. Min. Sci. **97**, 122 (2017).
4. *Zaitsev V., Radostin A., Pasternak E., Dyskin A.* // Non. Pr. Geophys. **24**, 543. (2017)

Why does the approach [1-4] ensure that the input and output of the model are consistent?

The same data for the Vosges Sandstone, but the model curves start from the minimum pressure



❑ Alternative description based on a model of a medium with crack-like defects characterized by independently introduced normal and shear compliance [1-4] (black line on the diff. models [3]) from previous slide. **There is no dependence on starting values of pressure (maximal or minimal) for differential model!**

An analytical solution of differential model in a parametric form via the parameter ***t = G / K***

$$\frac{K}{K_0}=\frac{t_0}{t}\left|\frac{R-t-\beta}{R-t_0-\beta}\right|^{\frac{4\beta+15+4R}{8R}}\left|\frac{R+t+\beta}{R+t_0+\beta}\right|^{-\frac{4\beta+15-4R}{8R}}$$

$$\boxed{G=Kt}$$

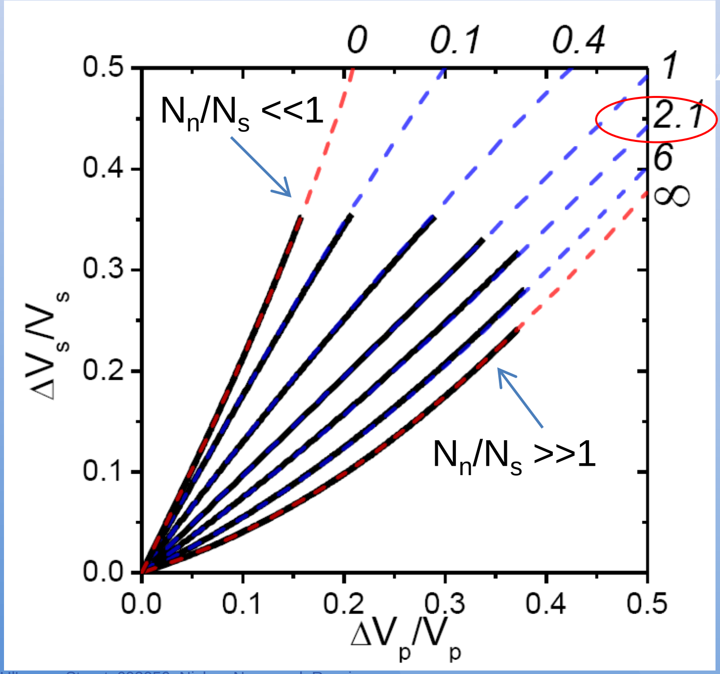
$$t=\frac{1}{\left(\dfrac{V_p}{V_s}\right)^2-\dfrac{4}{3}}$$

$$N_n-N_{n0}=\frac{135}{8R}\ln\left\{\left|\frac{R-t-\beta}{R-t_0-\beta}\right|^{-1}\left|\frac{R+t+\beta}{R+t_0+\beta}\right|\right\}$$

$$N_s-N_{s0}=(N_n-N_{n0})/q$$

$$\beta=27/4q-3/8$$

$$R=\sqrt{45/4+\beta^2}$$



In fact penny-shape cracks model corresponds to only one trajectory *q*~2.1 (!!!)

Predicted ranges of crack-induced variations in the S- and P-wave velocities for the entire possible range of the q-ratio, $0 \le q < \infty$. These predictions of the decoupled-compliances model can be considered as an analogue of Hashin-Strikman constrains.

(Hashin, Shtrikman, J Mech Phys Solids 1963)

Dashed lines are for differential model

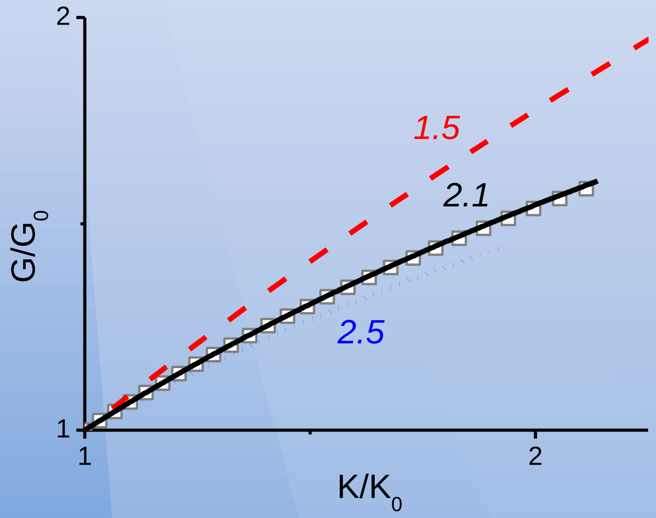
Solid lines are for no-interaction model and the same concentration of cracks

Comparison with penny-shape cracks model

$$\frac{K}{K_0}=\frac{t_0}{t}\left|\frac{t-3/2}{t_0-3/2}\right|^{10/9}\left|\frac{t+15/8}{t_0+15/8}\right|^{-1/9} \qquad t=G/K$$

Solution [Zimmerman, 1985] rewritten using parameter *t*

$$\Gamma-\Gamma_0=\ln\left\{\left|\frac{t+3/4}{t_0+3/4}\right|^{15/64}\left|\frac{t-3/2}{t_0-3/2}\right|^{-5/8}\left|\frac{t+15/8}{t_0+15/8}\right|^{5/128}\right\}$$



Penny-shape cracks (empty squares)

the solid line almost coinciding with the squares for penny-shape cracks corresponds to q=2.1;

dashed line is for q=1.5;

and dotted line is for q=2.5.

All the model curves correspond to the *same* range of variation in the parameter $t \in [0.75,1]$.

How should the correct model work so that the input and output match?

Experiment gives *G*(*p*) и *K*(*p*) => experimental line on the plane (*K*,*G*)
t* = *G* / *K (including of this ratio allows **exclude the pressure from analysis at this stage**)

$K^{\text{experim}}, G^{\text{experim}}$

$t_{initial}^{experim} \rightarrow t_{final}^{experim}$

input of the model is represented as $t_{initial}^{input} \rightarrow t_{final}^{input}$

For a chosen value of *q*-ratio, the model gives the *G*(*K*) dependence expressed as functions of parameter *t*

q-ratio should be corrected

Pressure dependence is a separate question:
The reasonable fit *t*(*p*) should be employed
t(*p*) =>
G(*p*) and *K*(*p*)

Predicted trajectory $(G,K)_{\textbf{predict}}$ and segment

$t_{initial}^{predict} \rightarrow t_{final}^{predict}$

Checking of consistency of $(G,K)_{predict}$ and $(G,K)_{experim}$, as well as length of the segment

(the last is often overlooked!)

$$(t_{initial}^{input} \rightarrow t_{final}^{input}) = (t_{initial}^{experim} \rightarrow t_{final}^{experim}) = (t_{initial}^{predict} \rightarrow t_{final}^{predict})$$

NO

YES => the model is correct
the input is consistent with output

46 Ul'yanov Street, 603950, Nizhny Novgorod, Russia

Resume: Correct procedure for the inversion of data

1. The first step is determining the properties of real defects by interpreting simultaneous changes in various moduli (for example, on the plane (G, K)). When constructing this relationship, pressure should be excluded. Fitting the ratio of compliance of the defects (which may differ greatly from 2.1) based on a model with explicitly entered compliances N_n and N_s allows one to correctly approximate the experimental trajectory on the plane (G, K).

2. As the second step, the pressure dependence for a chosen quantity is interpreted (this can be the defects' concentration, or a more convenient value that is uniquely determined by the concentration and measured experimentally, for example, the Poisson's ratio or the $t = G/K$ ratio). This requires an additional assumption about the type of relationship between the pressure and the selected parameter (for example, an exponential form).

The authors acknowledge the support of the Russian Foundation for Basic research (grant No 19-05-00536 a).

Thank you for your attention!

Any questions?

46 Ul'yanov Street, 603950, Nizhny Novgorod, Russia