Rapid distortion theory for homogeneous shear-driven MHD turbulence

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Motivation

Develop an analytical theory for description of statistical properties of MHD turbulence subjected to mean velocity shear and external magnetic field

Applications

- expanding solar wind
- astrophysical disks
- parametrisations for subgrid models

Rapid Distrortion Theory

$$T_D \frac{u}{l} \ll 1 \qquad T_D \frac{b^2}{ul} \ll 1$$

$T_D = S^{-1} = \left(\sqrt{S_{ij}S_{ij}}\right)^{-1}$ - distortion time scale

$u = \sqrt{\langle u_i u_i \rangle /3}$

$b = \sqrt{\langle b_i b_i \rangle /3}$ - characteristic velocity and field fluctuatuations

Governing equations

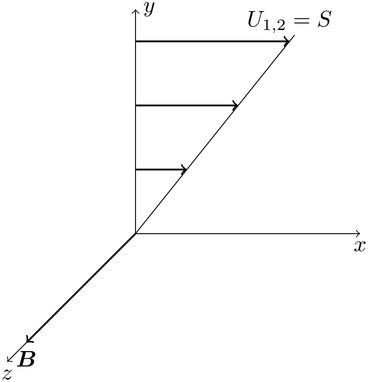
$$U_i = S_{ij}(t)\, x_j \qquad \boldsymbol{B} = \text{const}$$

$$\begin{cases} \dfrac{\partial u_i'}{\partial x_i} = 0 \\ \dfrac{\partial u_i'}{\partial t} + S_{jk} x_k \dfrac{\partial u_i'}{\partial x_j} + S_{ij} u_j' = -\dfrac{\partial \tilde{p}'}{\partial x_i} + B_j \dfrac{\partial b_i'}{\partial x_j} + (NL)_u \\ \dfrac{\partial b_i'}{\partial t} + S_{jk} x_k \dfrac{\partial b_i'}{\partial x_j} = B_j \dfrac{\partial u_i'}{\partial x_j} + S_{ij} b_j' + (NL)_b \end{cases}$$

$$(NL)_u = \frac{\partial}{\partial x_j} \left(b_i' b_j' - u_i' u_j' + \left\langle u_i' u_j' \right\rangle - \left\langle b_i' b_j' \right\rangle \right)$$

$$(NL)_b = \frac{\partial}{\partial x_j} \left(u_i' b_j' - u_j' b_i' + \left\langle u_j' b_i' \right\rangle - \left\langle u_i' b_j' \right\rangle \right)$$

Linear shear



$$S_{ij} = S\delta_{i1}\delta_{j2}$$

$$\boldsymbol{B} = (0, 0, B)$$

Rapid Distortion Theory
Rogallo transform

$$\begin{cases} \dfrac{\partial u_i'}{\partial x_i} = 0 \\ \dfrac{\partial u_i'}{\partial t} + S_{ij}x_j\dfrac{\partial u_i'}{\partial x_j} + u_j'S_{ij} = -\dfrac{\partial p}{\partial x_i} + B_j\dfrac{\partial b_i'}{\partial x_j} \\ \dfrac{\partial b_i'}{\partial t} + S_{ij}x_j\dfrac{\partial b_i'}{\partial x_j} = B_j\dfrac{\partial u_i'}{\partial x_j} + b_j'S_{ij} \end{cases}$$

Frame of reference moves along with the mean sheared velocity

$$x_i' = A_{ij}(t)x_j \quad t' = t$$

$$\hat{S} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{A}(t) = \begin{pmatrix} 1 & -St & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fourier transform $[\boldsymbol{u'}, \boldsymbol{b'}, p'] = \left[\hat{\boldsymbol{u}}(t), \hat{\boldsymbol{b}}(t), \hat{p}(t)\right] e^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}}$

$$\frac{dk_i(t)}{dt} = -S_{ji}k_j \qquad k_i(0) = k_{0i}$$

Equations for the evolution of the Fourier transform of the velocity and magnetic field

$$\begin{cases} \dfrac{d\hat{u}'_i}{dt} = -S_{il}\hat{u}'_l + \dfrac{2k_i k_n}{k^2} S_{nl}\hat{u}'_l + i\lambda k_l B_l \hat{b}'_i \\ \dfrac{d\hat{b}'_i}{dt} = S_{il}\hat{b}'_l + i k_l B_l \hat{u}'_i \end{cases}$$

$$\frac{dk_i(t)}{dt} = -S_{ji}k_j(t) \quad k_i(0) = K_i$$

$$P_{in} = \delta_{in} - \frac{k_i k_n}{k^2}$$

Vortical - isotropic in planes perpendicular to the axis of independence x_α

$$\Phi_{ij}^{vor} = \frac{E(k)}{2\pi k}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)(1 - \delta_{i\alpha})(1 - \delta_{j\alpha})$$

Jetal - fluctuations in the direction of the axis of independence x_α

$$\Phi_{ij}^{jet} = \frac{E(k)}{2\pi k}\delta_{i\alpha}\delta_{j\alpha}$$

where $\Phi_{ij} = \left\langle \hat{u}_i \hat{u}_j^* \right\rangle$, $\int_{k=0}^{\infty} E(k)dk = \frac{q_0^2}{2}$

Solutions for fluctuations of velocity and magnetic field

$$\hat{u}_1(\beta) = \hat{u}_1^0 \cos(\gamma\beta\sin\phi) - \hat{u}_2^0 \frac{1}{\gamma\sin\phi}\sin(\gamma\beta\sin\phi) + i\hat{b}_1^0 \sin(\gamma\beta\sin\phi)$$

$$\hat{u}_2(\beta) = \hat{u}_2^0 \cos(\gamma\beta\sin\phi) + i\hat{b}_2^0 \sin(\gamma\beta\sin\phi)$$

$$\hat{u}_3(\beta) = \hat{u}_3^0 \cos(\gamma\beta\sin\phi) + i\hat{b}_3^0 \sin(\gamma\beta\sin\phi)$$

$$\hat{b}_1(\beta) = \hat{b}_1^0 \cos(\gamma\beta\sin\phi) + \hat{b}_2^0 \frac{1}{\gamma\sin\phi}\sin(\gamma\beta\sin\phi) + i\hat{u}_1^0 \sin(\gamma\beta\sin\phi)$$

$$\hat{b}_2(\beta) = \hat{b}_2^0 \cos(\gamma\beta\sin\phi) + i\hat{u}_2^0 \sin(\gamma\beta\sin\phi)$$

$$\hat{b}_3(\beta) = \hat{b}_3^0 \cos(\gamma\beta\sin\phi) + i\hat{u}_3^0 \sin(\gamma\beta\sin\phi)$$

$$\beta = St \qquad \gamma = \frac{B}{Sl}$$

$$k_2 = k\cos\phi \quad k_3 = k\sin\phi$$

2D-3C turbulence. Case $k_1 = 0$

Solutions for fluctuations of velocity and magnetic field

$$
\begin{aligned}
\hat{u}_1(\beta) &= \hat{u}_1^0 \cos(\gamma\beta\sin\phi) - \hat{u}_2^0 \frac{1}{\gamma\sin\phi}\sin(\gamma\beta\sin\phi) + i\hat{b}_1^0 \sin(\gamma\beta\sin\phi) \\
\hat{u}_2(\beta) &= \hat{u}_2^0 \cos(\gamma\beta\sin\phi) + i\hat{b}_2^0 \sin(\gamma\beta\sin\phi) \\
\hat{u}_3(\beta) &= \hat{u}_3^0 \cos(\gamma\beta\sin\phi) + i\hat{b}_3^0 \sin(\gamma\beta\sin\phi)
\end{aligned}
$$

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\end{aligned}
$$

$$
\beta = St \qquad \gamma = \frac{B}{Sl}
$$

$$
k_2 = k\cos\phi \quad k_3 = k\sin\phi
$$

2D-3C turbulence. Case $k_1 = 0$

Second order moments evolution

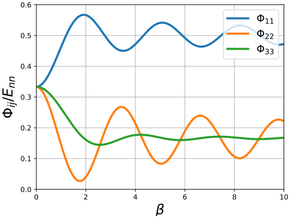
$$\Phi_{11}(\beta) = \frac{1 + 2/\gamma^2 + \left(1 - 2/\gamma^2\right) J_0\left(2\gamma\beta\right)}{6}$$
$$\Phi_{22}(\beta) = \frac{1 - 2J_2\left(2\gamma\beta\right) + J_1\left(2\gamma\beta\right)/(\gamma\beta)}{6}$$
$$\Phi_{33}(\beta) = \frac{1 + J_1\left(2\gamma\beta\right)/(\gamma\beta)}{6}$$

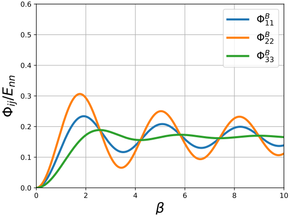
$$\Phi^B_{11}(\beta) = \frac{1 - J_0\left(2\gamma\beta\right)}{6}$$
$$\Phi^B_{22}(\beta) = \frac{1 + 2J_2\left(2\gamma\beta\right) - J_1\left(2\gamma\beta\right)/(\gamma\beta)}{6}$$
$$\Phi^B_{33}(\beta) = \frac{1 - J_1\left(2\gamma\beta\right)/(\gamma\beta)}{6}$$

$\beta = St \qquad \gamma = \dfrac{B}{Sl} \quad J_n(x)$ - Bessel functions of the 1st kind

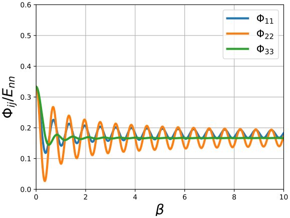
Second order moments evolution

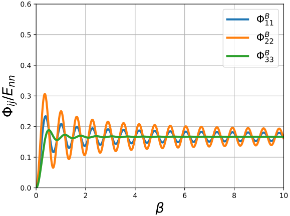
$$\frac{B}{Sl} = 1$$





$$\frac{B}{Sl} = 5$$





Future plans

- rotating flow
- magnetic shear
- geostrophic turbulence on beta-plane