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GRAD-SHAFRANOV RECONSTRUCTION OF THE IN-PLANE MAGNETIC FIELD POTENTIAL IN THE X-POINT VICINITY: BOUNDARY-LAYER APPROXIMATION

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This work is funded by Austrian Science Fund (FWF): P27012-N27 and Russian Science Foundation (RSF): 18-47-05001

FWF Der Wissenschaftsfonds.

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OUTLINE

1. EMHD approximation.
2. The Grad-Shafranov equation for magnetic potential.
3. Boundary layer approximation.
4. Benchmark reconstruction of the magnetic potential by means of different techniques, resting upon PIC simulations data.
5. Comparison of the results in terms of relative error of reconstruction.
6. Conclusion: the approximate solution of the Grad-Shafranov equation (solution of the well-posed problem) is more accurate than its “exact” solution (solution of the ill-posed problem).

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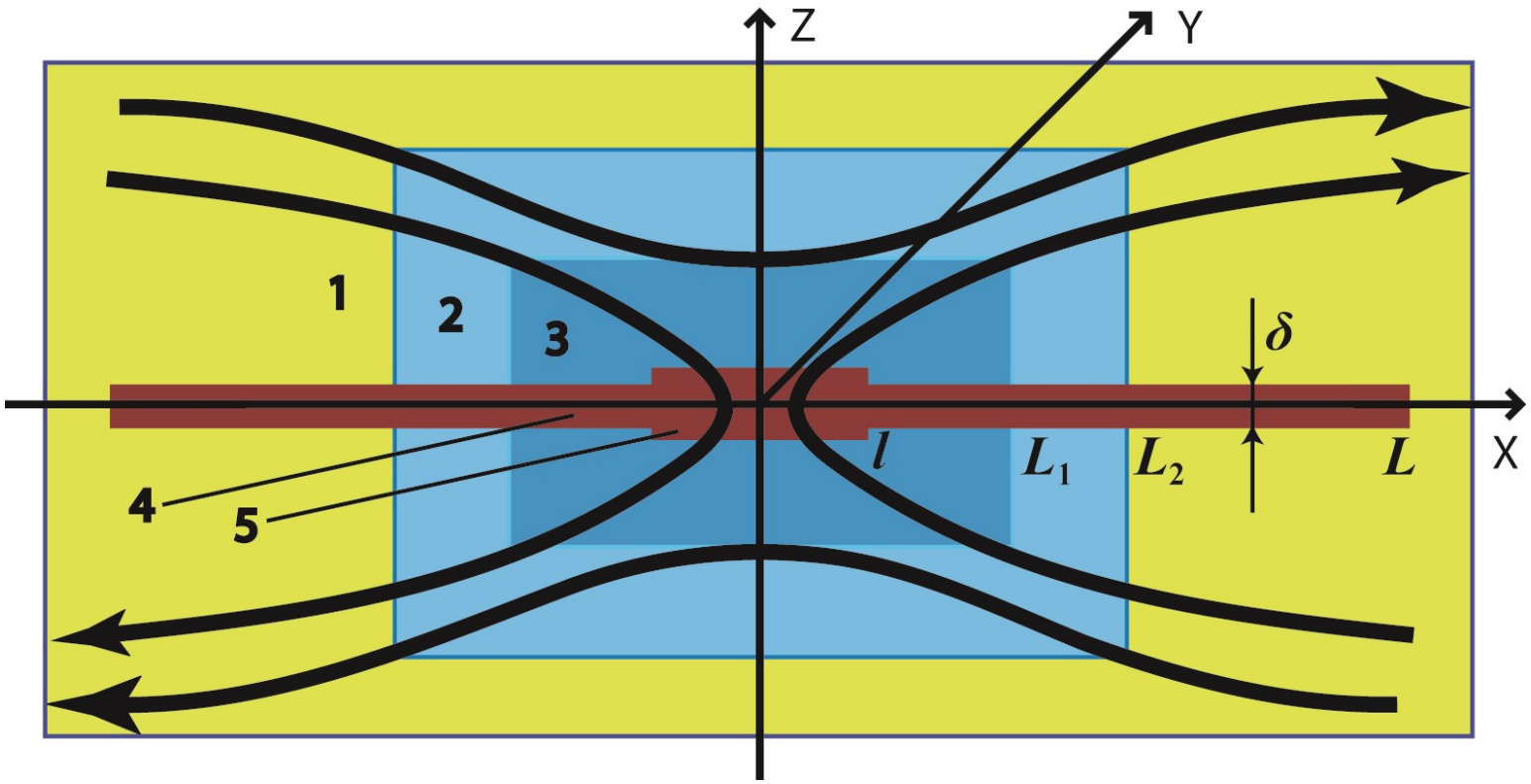
THE REFERENCE SYSTEM

Sketch of the magnetic configuration and adopted reference system



Here, numbers 1, 2, and 3 mark MHD, HMHD, and EMHD regions, respectively, and 4 and 5 show external and internal EDR.

After [*Korovinskiy et al. (2011), JGR 116, A05219*]

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EMHD APPROXIMATION

The reconstruction problem for steady symmetrical two-dimensional (2d) magnetic reconnection in the X-point vicinity is addressed in terms of the EMHD approximation,

$$m_e(\mathbf{V}_e \cdot \nabla)\mathbf{V}_e = -\frac{1}{n_e}\nabla \cdot \hat{P}_e - (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}),$$

$$\nabla \times \mathbf{B} = -n_e \mathbf{V}_e,$$

$$\nabla \times \mathbf{E} = 0,$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \cdot \mathbf{E} = \frac{c^2}{V_A^2}(n_i - n_e),$$

$$\nabla \cdot (n_{i,e}\mathbf{V}_{i,e}) = 0.$$

Here, all notations are conventional and all quantities are normalized.

Normalization constants: $\left\{m_i, d_i = \frac{c}{\omega_i}, n_0, B_0, V_A = \frac{B_0}{\sqrt{4\pi n_0 m_i}}, E_A = \frac{1}{c}B_0 V_A, P_0 = \frac{B_0^2}{4\pi}\right\}$.

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EQUATION FOR MAGNETIC POTENTIAL

Introducing the magnetic potential $A(x,z)$ of the in-plane magnetic field, $\mathbf{B}_\perp = [\nabla A \times \mathbf{e}_y]$, and substituting this definition to the Ampère's law, we derive the equation for A:

$$\Delta_\perp A = n_e(x,z)V_{ey}(x,z), \qquad \text{(I)}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial z^2$ stands for the in-plane Laplace operator.

Thus, we arrive at the two-dimensional Poisson's equation.

Next, one can perform the variables transformation $(x,z) \rightarrow (A, B_y)$, where B_y appears to be a stream function of the electron in-plane flow.

In a simplest case $\{n_e = const,\ T_e = const\}$ the right-hand side of Eq.(I) turns to the function on A only with $V_{ey} = V_{ey}(A)$ [*Uzdensky and Kulsrud (2006), PoP 13, 062305; Korovinskiy et al. (2008), JGR 113, A04205*].

In a more general case $\{n_e = n_e(A), T_e = T_e(A)\}$, one gets the representation $V_{ey} = V_{ey1}(A) + V_{ey2}(A)B_y^2$, hence the right-hand side of Eq.(I) acquires also the 2nd term, depending on both variables [*Korovinskiy et al. (2011), JGR 116, A05219; Sonnerup et al. (2016), JGR 121, 4279*].

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GRAD-SHAFRANOV EQUATION

So, assuming $\{n_e = n_e(A),\ \ T_e = T_e(A)\}$, we have

$$\Delta_\perp A = \frac{dL(A)}{dA} + O\left(A, B_y^2\right), \qquad \text{(II)}$$

where

$$L(A) = \int_{-\infty}^{A} n_e(A') V_{ey1}(A') dA'. \qquad \text{(III)}$$

In a small vicinity of the X-point, where $B_y = 0$ due to the symmetry condition, one can neglect the second term in the right-hand side of Eq.(II), arriving at the well-known Grad-Shafranov equation.

The quantity V_{ey1} is the major part of V_{ey}; it is calculated from the boundary conditions, fixed as magnetoplasma parameters at some curve $\hat{S}$ (the satellite trajectory), as well as $n_e(A)$ and A itself,

$$A = \int_{\hat{S}} (B_z dx - B_x dz) + A_0, \qquad \text{(IV)}$$

$$V_{ey1}(A) = \left[V_{ey} + \frac{B_y^2}{2n_e} \frac{d}{dA} \ln(n_e) \right]_{\hat{S}}.$$

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BOUNDARY LAYER APPROXIMATION

Small reconnection rate, which we denote ε, dictates stretched magnetoplasma configuration, where $\frac{\partial}{\partial x} \sim \varepsilon \frac{\partial}{\partial z}$, hence

$$\Delta_\perp A = \frac{\partial^2 A}{\partial z^2} + O(\varepsilon^2).$$

Neglecting the term $\sim\varepsilon^2$, we obtain the boundary layer (BL) approximation for the Grad-Shafranov equation for magnetic potential,

$$\frac{\partial^2 A}{\partial z^2} = \frac{dL(A)}{dA}. \qquad \text{(V)}$$

Multiplying Eq. (V) by $\partial A/\partial z$ and integrating it over z, we derive the solution

$$z(A) = z_0 \pm \frac{1}{\sqrt{2}} \int_{A_0}^{A} \frac{dA'}{\sqrt{L(A') - L(A_0) + B_{x0}^2/2}} \qquad \text{(VI)}$$

where subscript 0 marks the initial values (values at the satellite trajectory).

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THE RECONSTRUCTION PROBLEM

So, the reconstruction problem and its reductions are formulated as follows,

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} = n_e(x,z)\mathrm{V}_{ey}(x,z) \qquad \text{(1) Poisson}$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} = n_e(A)\mathrm{V}_{ey}(A) \qquad \text{(2) G.-Sh.}$$

$$\frac{\partial^2 A}{\partial z^2} = \frac{dL(A)}{dA} \qquad \text{(3) BL G.-Sh.}$$

$$z(A) = z_0 \pm \frac{1}{\sqrt{2}} \int_{A_0}^{A} \frac{dA'}{\sqrt{L(A') - L(A_0) + B_{x0}^2/2}} \qquad \text{(4) BL G.-Sh.}$$

Notably, despite Eq. 4 is derived by integrating Eq. 3, these two equations are not identical from the computational point of view.

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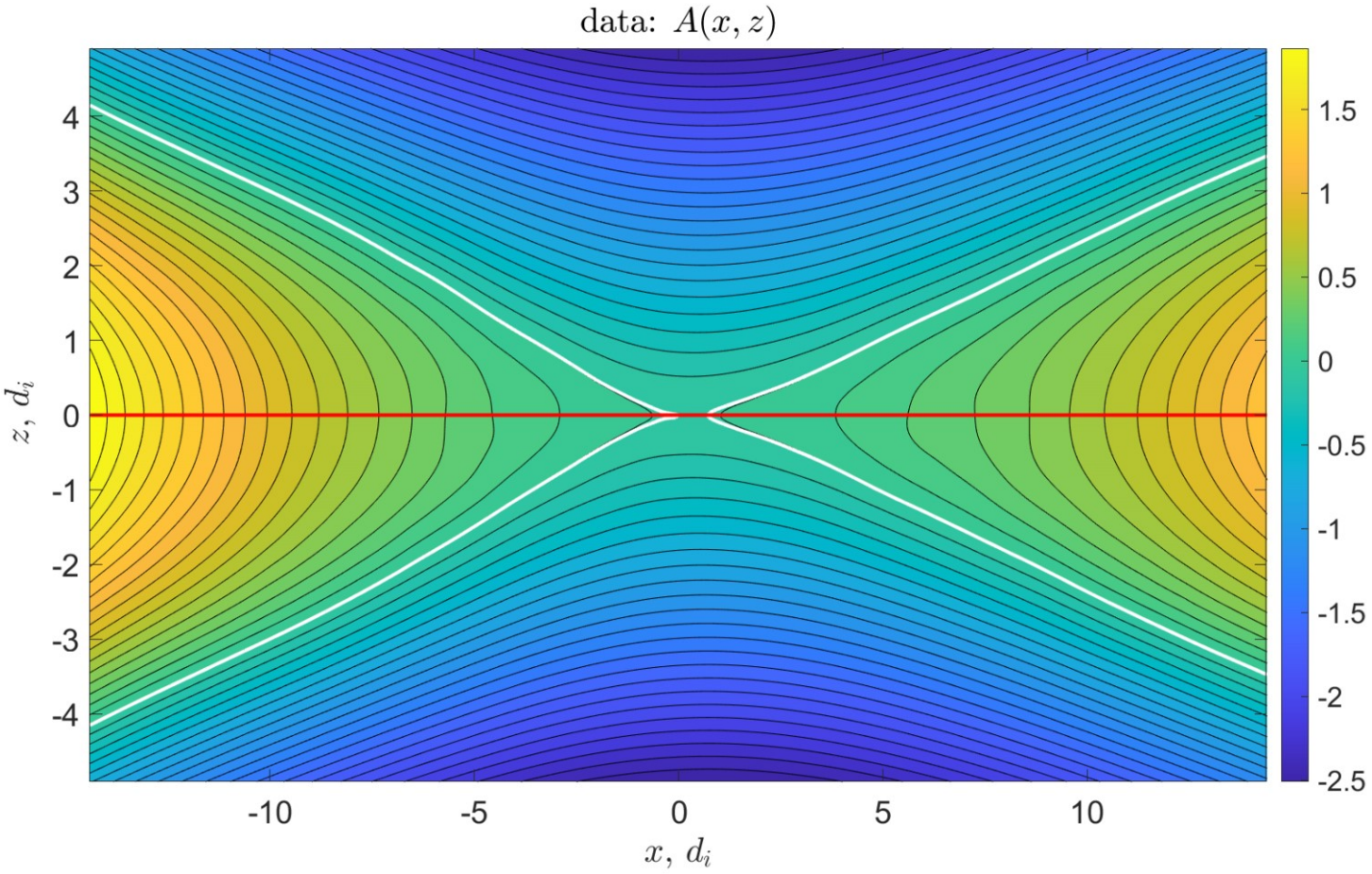
TEST CONFIGURATION

Magnetic configuration, modeling the steady symmetrical reconnection, is obtained by means of two-dimensional PIC simulations with $m_i/m_e = 256$. White curves show the magnetic separatrices, and red line imitates the satellite trajectory.



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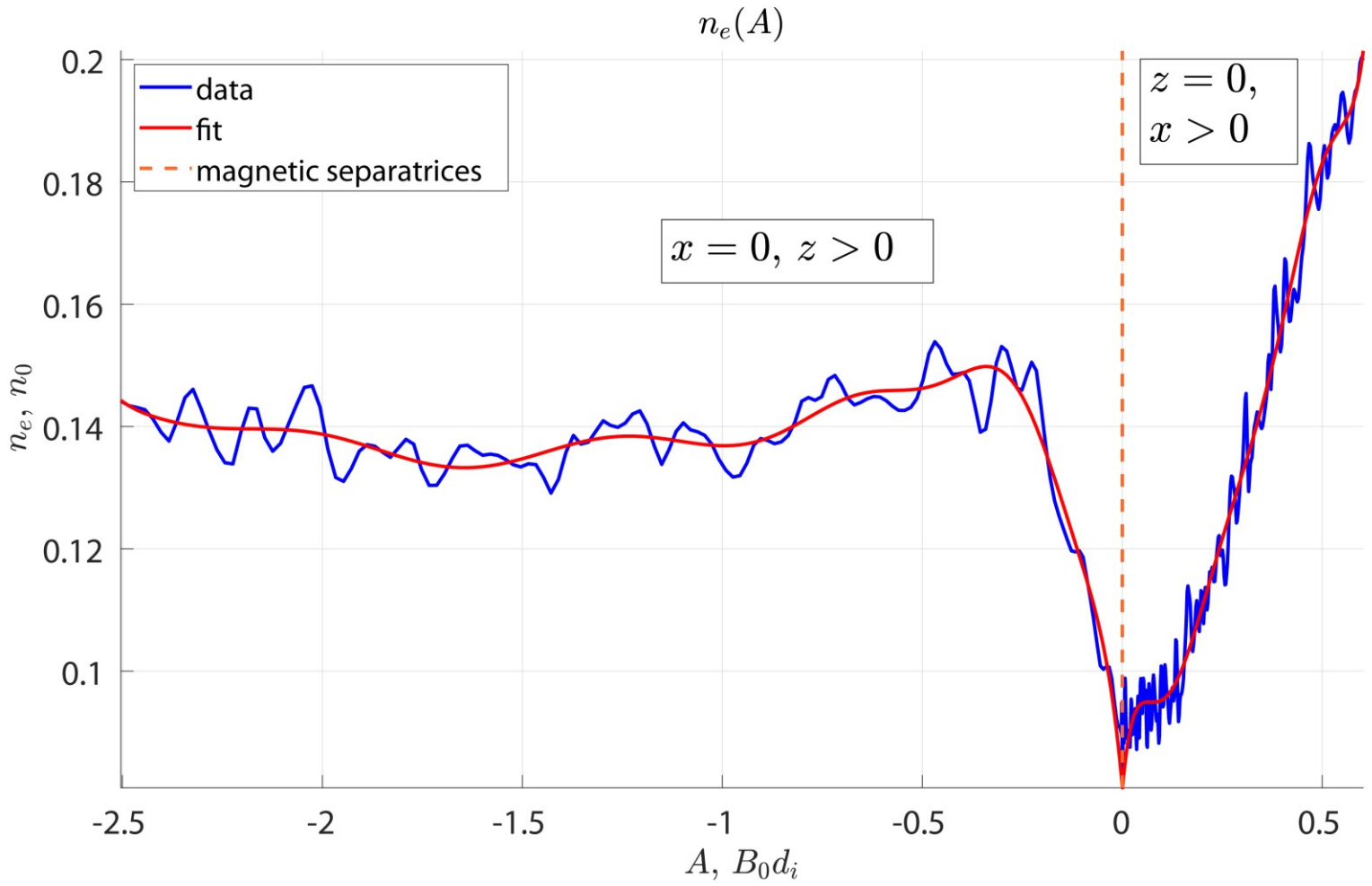
TEST FUNCTIONS: $n_e(A)$

Test functions are obtained by merging two parts, $f(A) = f_1(A < 0) + f_2(A > 0)$, where f_1 is evaluated at the cross-section $\{x = 0, z > 0\}$, and f_2 at $\{z = 0, x > 0\}$.



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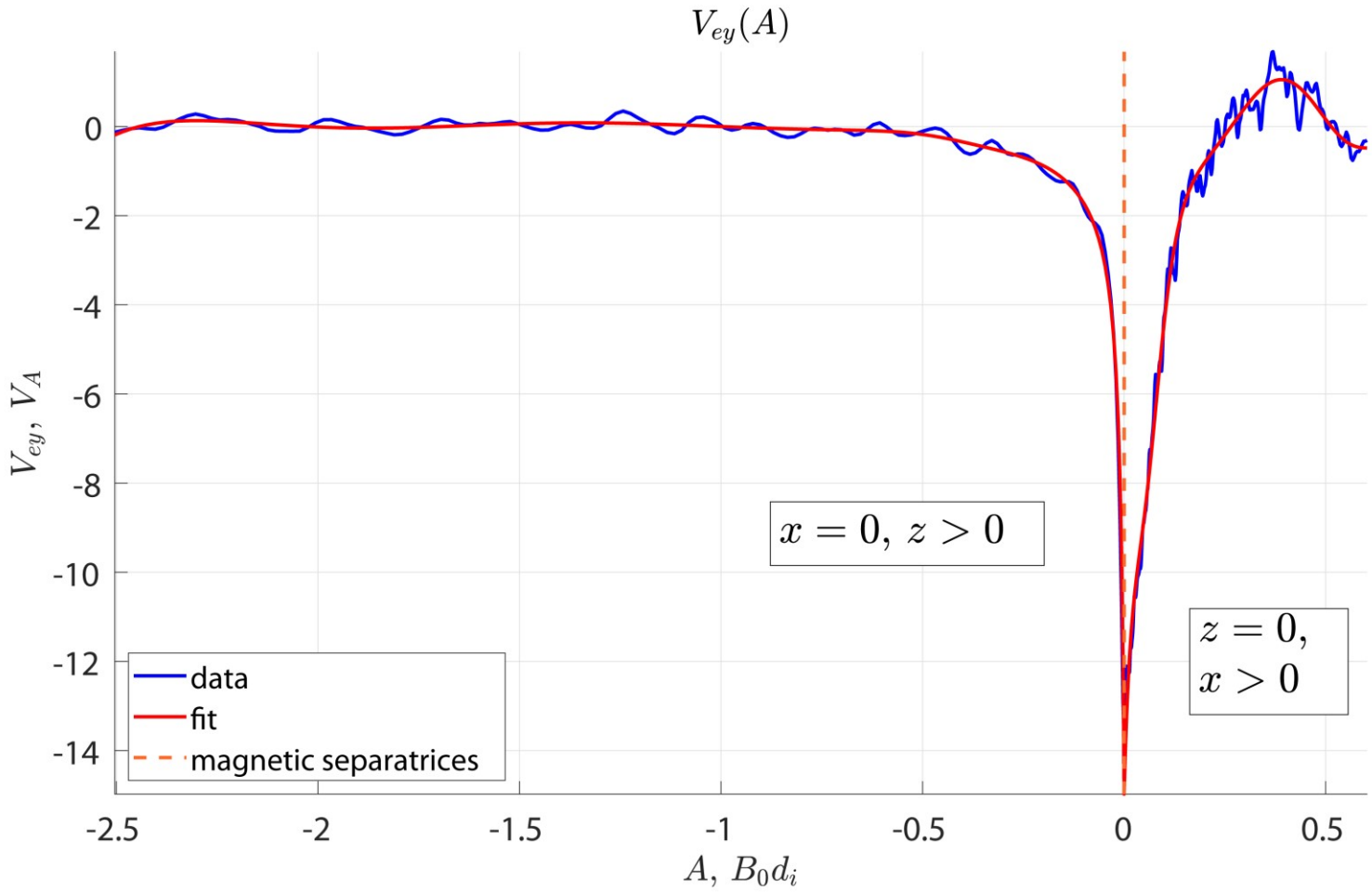
TEST FUNCTIONS: $V_{ey}(A)$

Since at the x axis $B_y = 0$ due to the symmetry condition, we have $\{V_{ey1}(A) = V_{ey}(A), V_{ey2} = 0\}$, hence Eq. (II) turns to the G.-Sh. Eq. (2) exactly.



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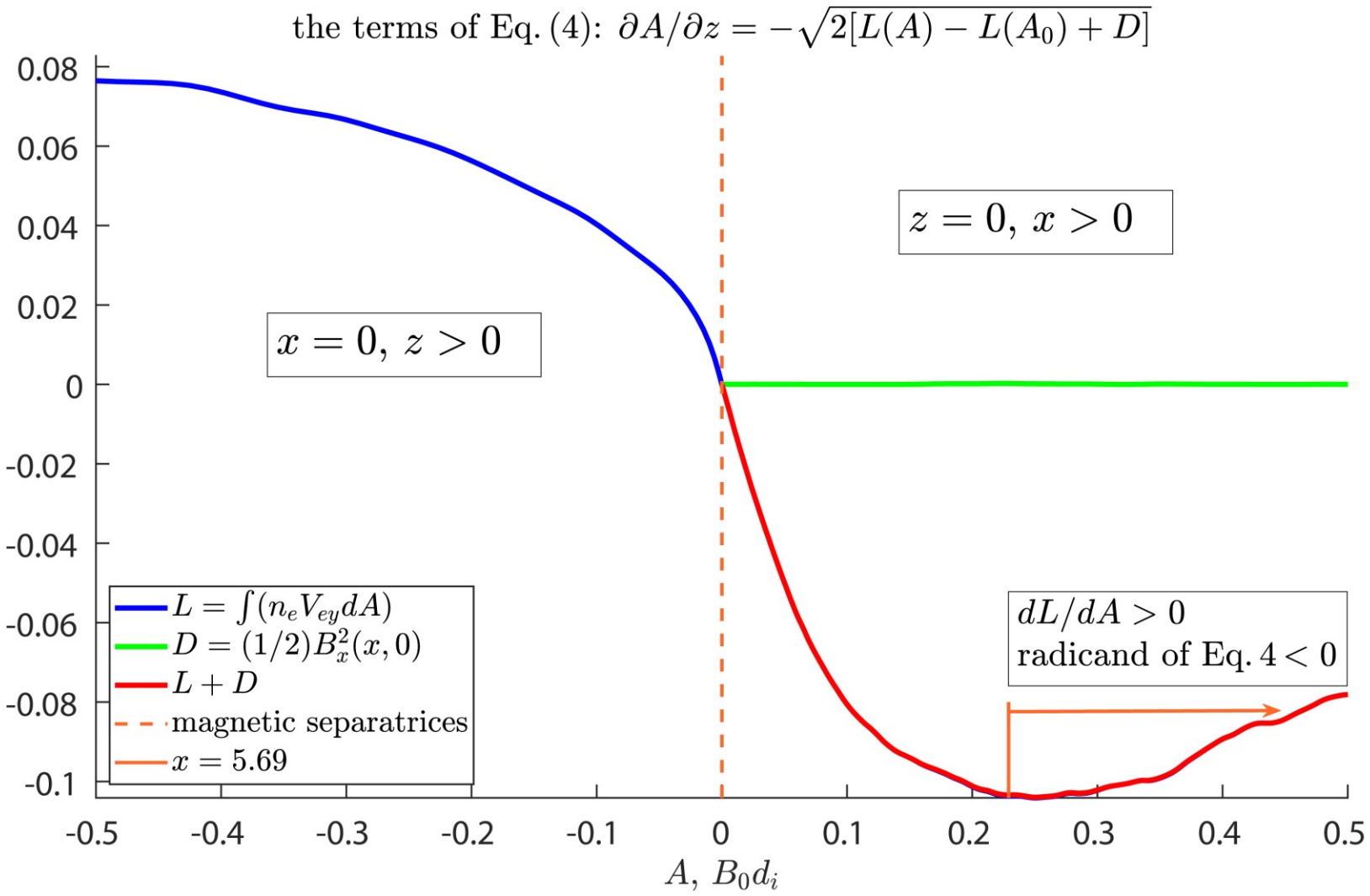
TEST FUNCTIONS: $L(A)$



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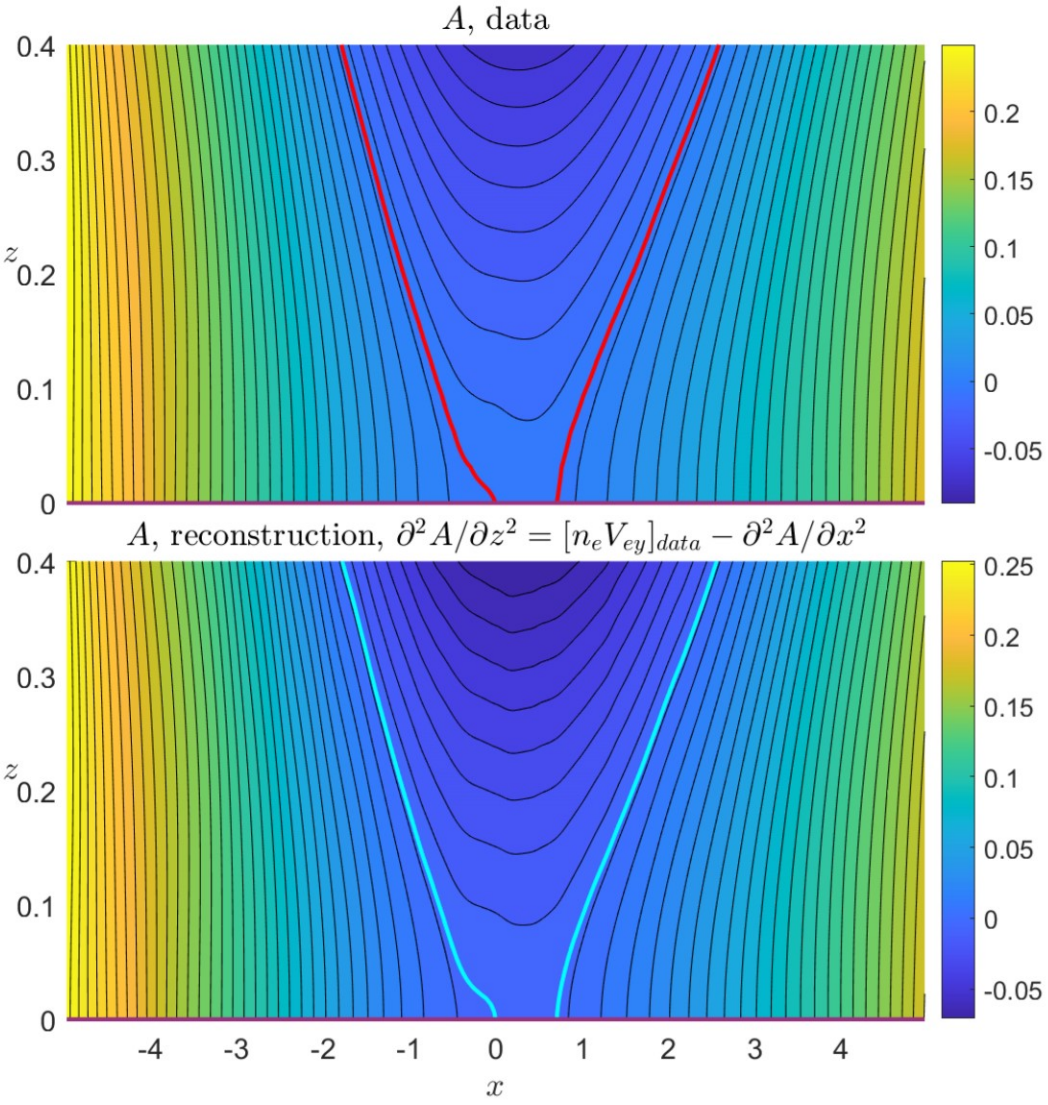
RECONSTRUCTION: POISSON (1)



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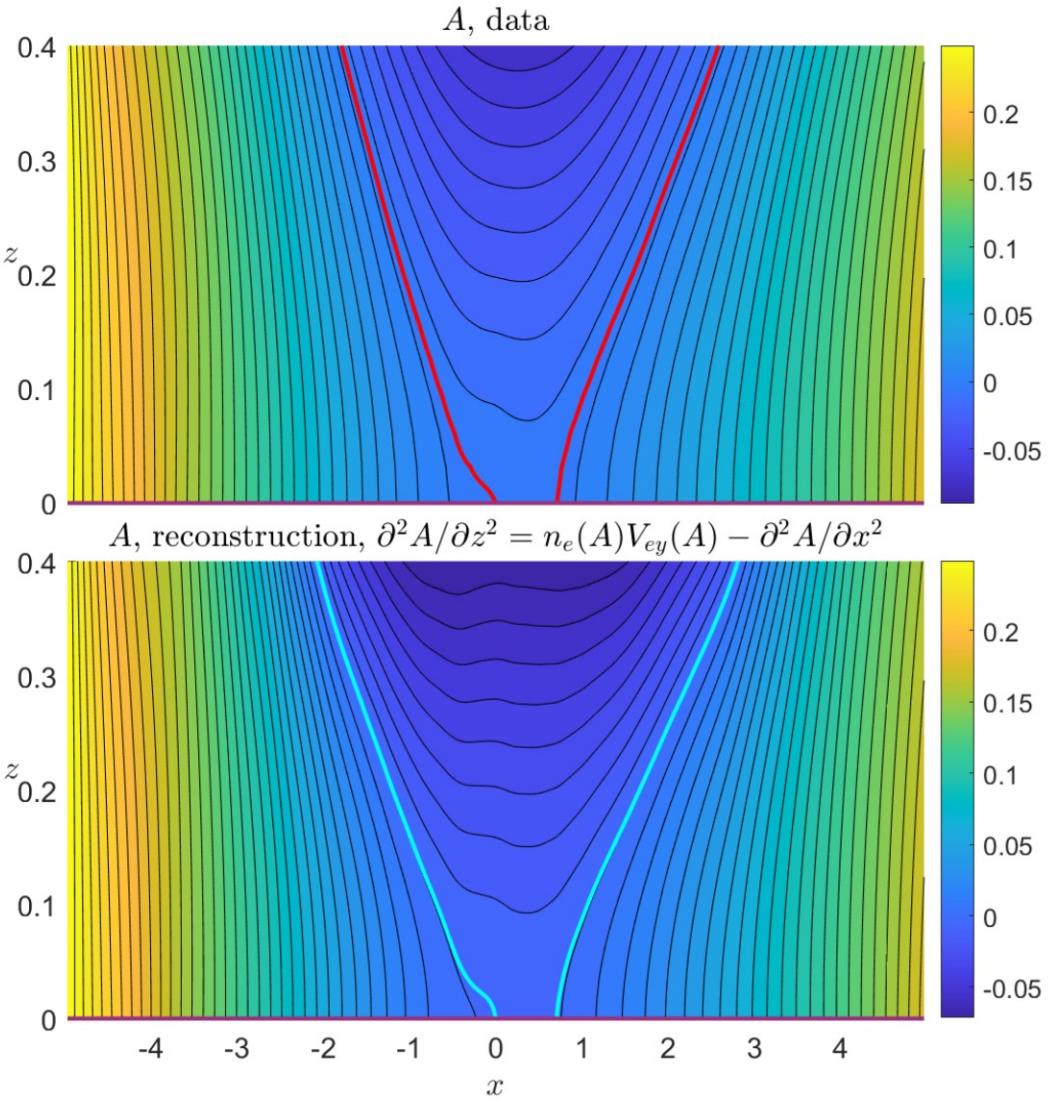
RECONSTRUCTION: GRAD-SHAFRANOV (2)



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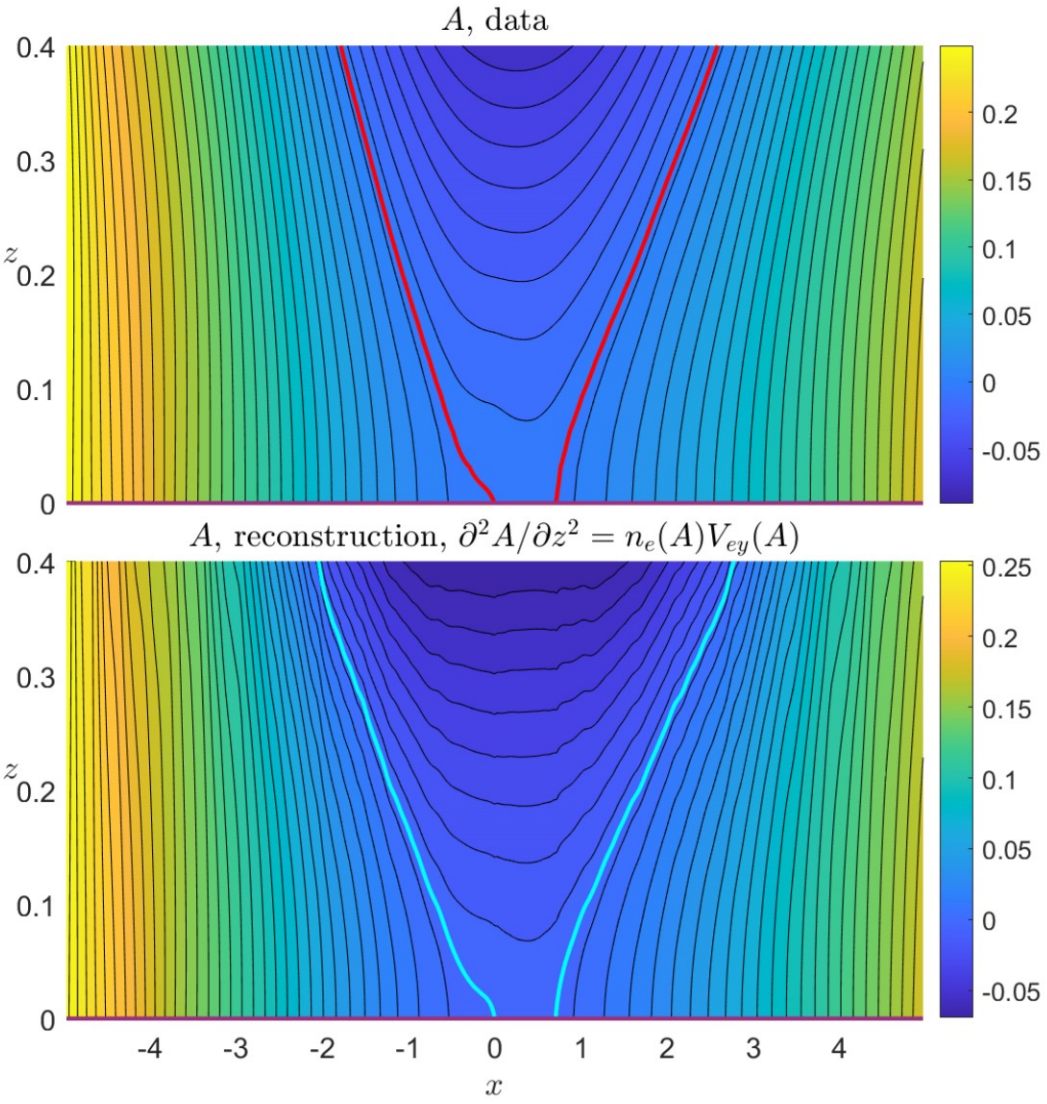
RECONSTRUCTION: BL G.-SH. (3)



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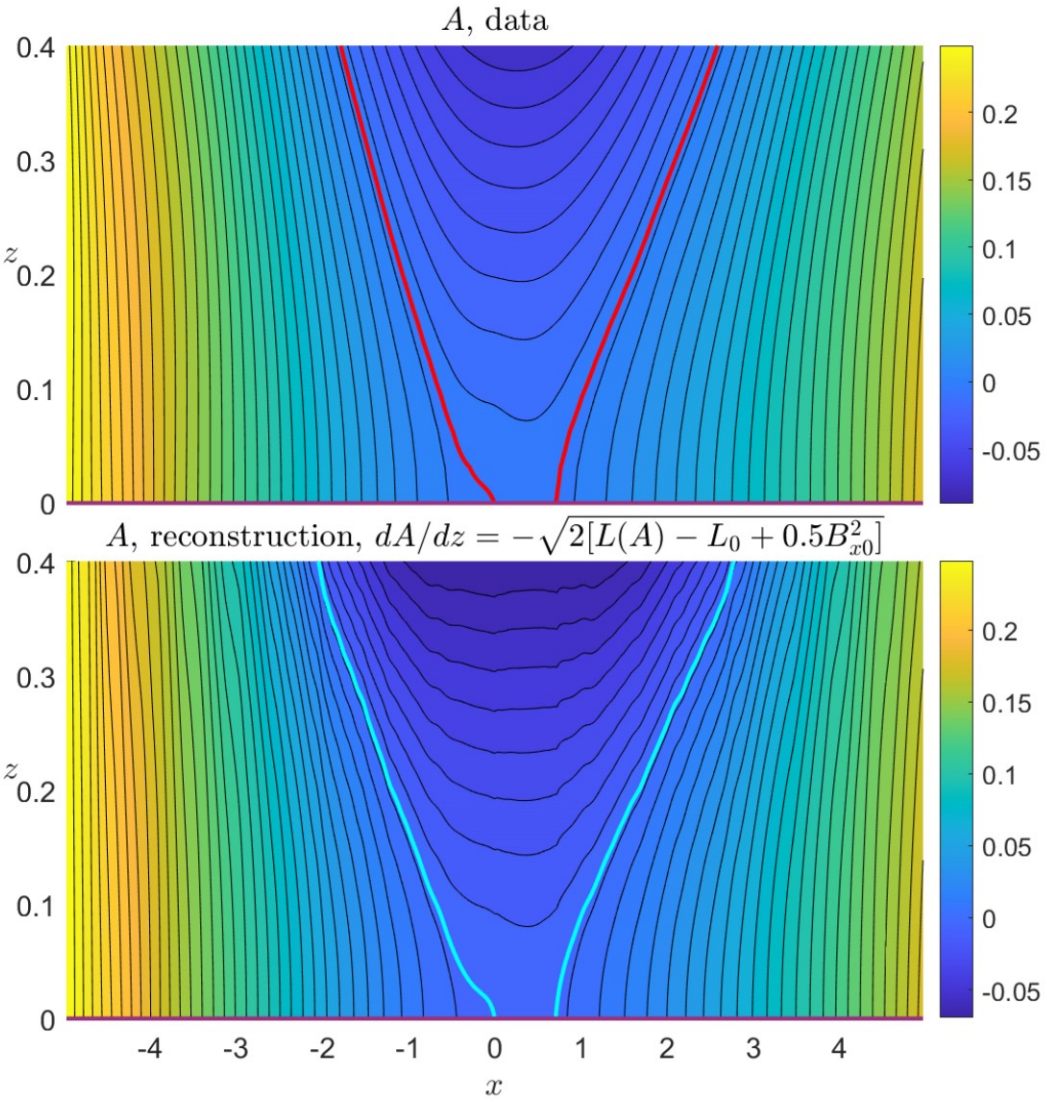
RECONSTRUCTION: BL G.-SH. (4)



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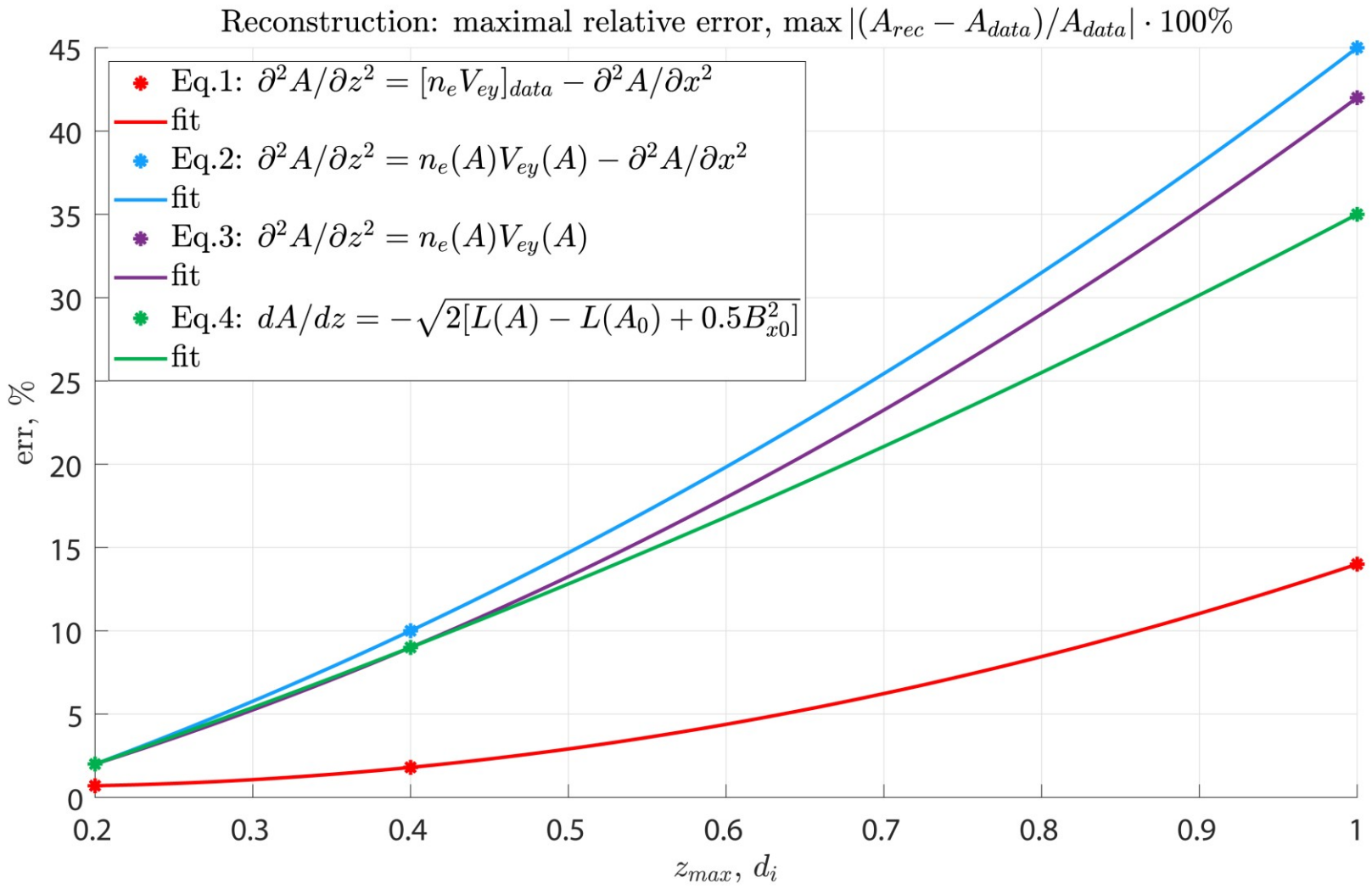
RECONSTRUCTION: ERROR



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RECONSTRUCTION ERROR: SUMMARY

1. The reconstruction error is growing rather fast with a distance from the initial profile, amounting to $\approx 2\%$ at $z = 0.2d_i$ and $\approx 10\%$ at $z = 0.4d_i$.

2. Solution of the test problem, Eq.(1), where the right-hand side of the Poisson's equation is taken from PIC simulations data at each step of integration, shows also the growing error, though several times less. Inaccuracy of this solution has three sources:

   a) inaccuracy of the numerical calculation of the derivative $\frac{\partial^2 A}{\partial x^2}$;

   b) artificial smoothing (Savitzky-Golay filter);

   c) nonideal stationary state of the magnetic configuration.

3. The worst accuracy is demonstrated by the "exact" solution of the Grad-Shafranov equation (2), where the dependence on B_y in the right-hand side of the Poisson's equation is assumed to vanish.

4. The boundary layer approximation (Eq.3,4) improves the solution; at large distances the best reconstruction accuracy is achieved by the solution of the BL G.-Sh. problem, cast in a form of the 1st order ODE (Eq.4). *Notably, Eq. 3 and 4 do not contain the term $\partial^2 A/\partial x^2$, hence no artificial smoothing is applied.*

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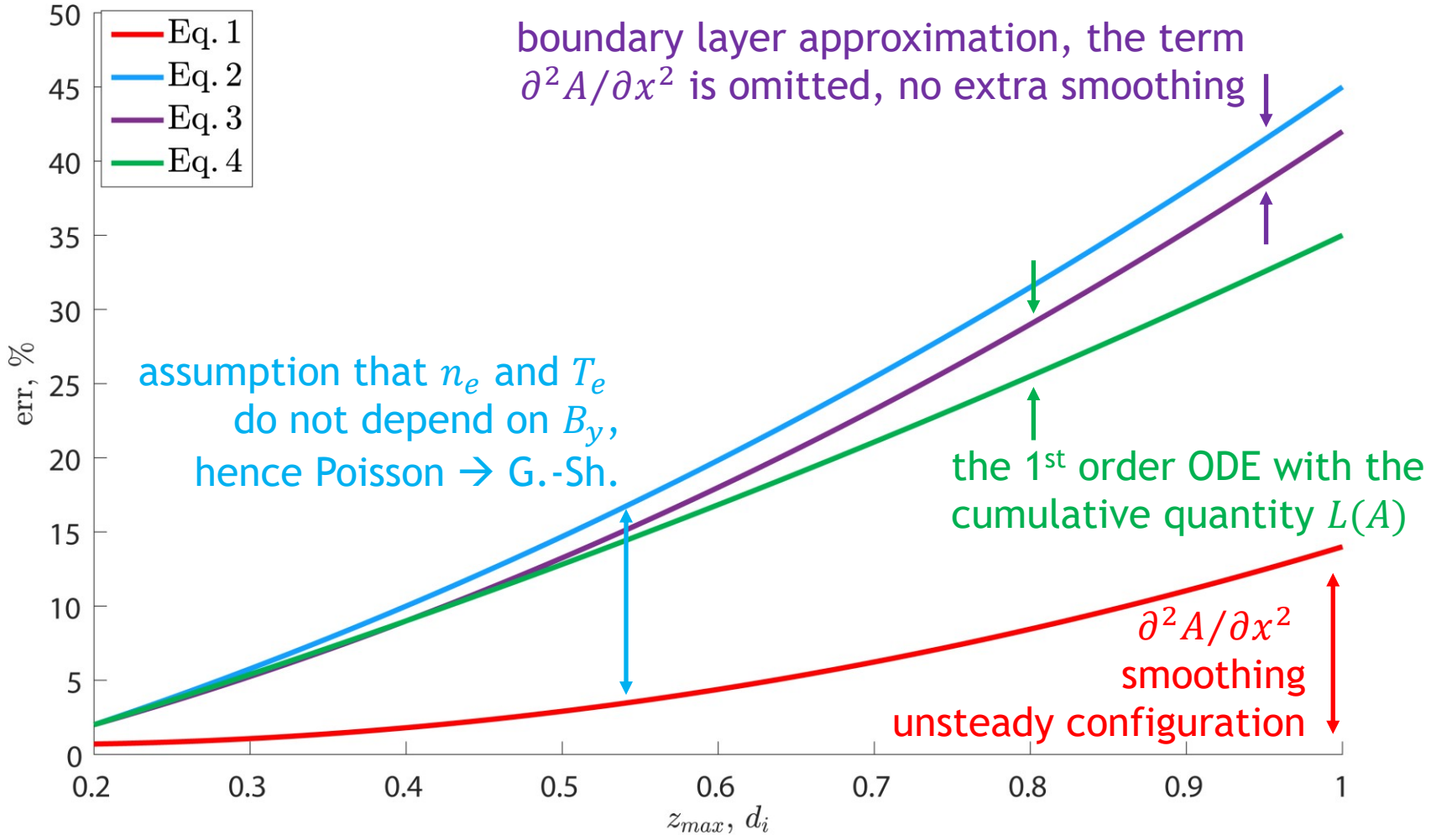
RECONSTRUCTION ERROR SOURCES



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SUMMARY AND CONCLUSIONS

The reconstruction problem for steady symmetrical 2d magnetic reconnection is addressed in a frame of EMHD approximation, yielding the Poisson's equation for the magnetic potential $A(x,z)$ of the in-plane magnetic field, $\Delta A = n_e V_{ey}$.

Performing the variables transformation $(x,z) \to (A, B_y)$, and assuming $n_e = n_e(A)$ and $T_e = T_e(A)$, the Poisson's equation turns to the Grad-Shafranov equation with small extra term $\sim O(B_y^2)$. In our particular case this term does vanish. With boundary conditions fixed at unclosed curve, the problem is ill-posed.

Since reconnection rate $\varepsilon \ll 1$, one can make use of the boundary layer approximation, assuming $\Delta A = \partial^2 A/\partial z^2 + O(\varepsilon^2)$ and neglecting the term $\sim \varepsilon^2$. This way, the equation for magnetic potential turns to the 2nd order ODE, which can be integrated one time analytically. The problem is well-posed.

The benchmark reconstruction of $A(x,z)$, obtained from PIC simulations, has shown that the main contribution for inaccuracy arises from replacing the Poisson's equation by the Grad-Shafranov one. This simplification imposes a severe restriction on the reachable cross-size of the reconstructed region, $z_{max} \sim 0.5 d_i$. At the same time, the longitudinal size may reach the value of several d_i.

Boundary layer approximation improves the accuracy, since the problem becomes well-posed; at large distances 1st order ODE form is preferable.