Gravity and magnetic data analysis based on Poisson wavelet-transforms

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Moscow, 2020

Continuous Wavelet-Transform

- Continuous wavelet-transform is a convolution of the analyzed function $g(\xi)$ with the function $\psi_{h,x}(\xi)$:

$$W(h,x) = \int_{-\infty}^{\infty} g(\xi)\overline{\psi_{h,x}(\xi)}\,d\xi$$

- Line under $\psi_{h,x}(\xi)$ means its complex conjugation. Function $W(h,x)$ is wavelet-spectrum of function $g(\xi)$. Function $\psi_{h,x}(\xi)$ take out the base (mother) wavelet $\psi_0(\xi)$:

$$\psi_{h,x}(\xi) = \frac{1}{\sqrt{h}}\psi_0\left(\frac{\xi - x}{h}\right)$$

- Parameters h – **scale** of wavelet-transform. It takes values from zero, but not including it, to infinity (**frequency** equivalent); x – **shift** parameter that determines the position of the wavelet on the axis $O\xi$ (axis Ox).

- Function $\psi_0(\xi)$ must meet certain requirements, in particular:

$$\int_{-\infty}^{\infty} \psi_0(\xi)\,d\xi = 0, \qquad \int_{-\infty}^{\infty} \left|\psi_0(\xi)\right|^2 d\xi < \infty$$

Continuous Wavelet-Transform for 2D signals

The continuous wavelet transform of two-dimensional signals can be written as:

$$W(a_x, a_y, x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\psi_{a_x,a_y,x,y}(\xi,\eta)\,d\xi d\eta$$

where

$$\psi_{a_x,a_y,x,y}(\xi,\eta) = \frac{1}{\sqrt{a_x}\sqrt{a_y}}\psi_0\left(\frac{\xi - x}{a_x}, \frac{\eta - y}{a_y}\right)$$

Wavelets must meet the following conditions:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_0(\xi,\eta)\,d\xi d\eta = 0, \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\psi_0(\xi)\right|^2 d\xi d\eta < \infty$$

For convenience, the scale coefficients a_x and a_y can be taken equal and further denoted as h:

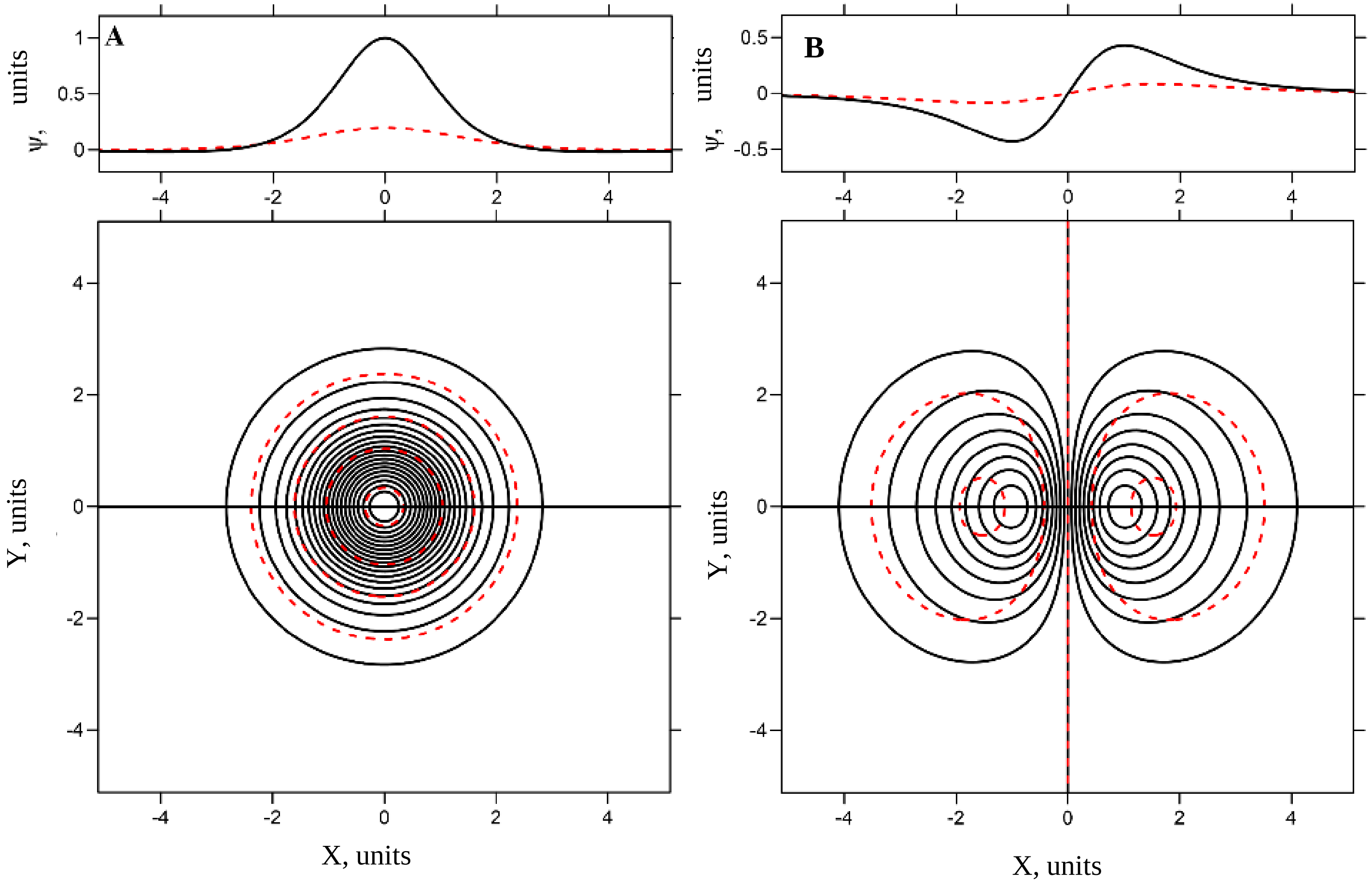
$$\boxed{\psi_{h,x,y}(\xi,\eta) = \frac{1}{h}\psi_0\left(\frac{\xi - x}{h}, \frac{\eta - y}{h}\right)}$$

As a result of the wavelet-transform with the represented wavelet, the volume wavelet-spectrum $W(h,x,y)$ can be calculated.

Poisson wavelets

- For gravity and magnetic fields anomalies analysis, it is possible to use wavelets based on the Poisson kernel as its derivatives:

$$K^{nx,my,kz}(x,y,z)=\frac{\partial^{n+m+k}}{\partial^n x\,\partial^m y\,\partial^k z}\left(\frac{z}{\sqrt{(x^2+y^2+z^2)^3}}\right)$$



1-st order Poisson wavelets. A – corresponding to the vertical derivative; B – corresponding to the horizontal derivative. Solid line- h = 1, red dashed line - h = 2.

1-st order Poisson wavelets

The Poisson kernel corresponding to the calculation of the first vertical derivative at height *z* is:

$$K(x,y,z)=\frac{1}{2\pi}\frac{2z^2-x^2-y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^5}$$

Equating $z = 1$ and omitting the multiplier $(1/2\pi)$ we can write a basic wavelet:

$$\psi_0^z(\xi,\eta)=\frac{2-\xi^2-\eta^2}{\left(\sqrt{\xi^2+\eta^2+1)}\right)^5}$$

It can be used to construct a 1st-order wavelet corresponding to the second vertical derivative of the potential:

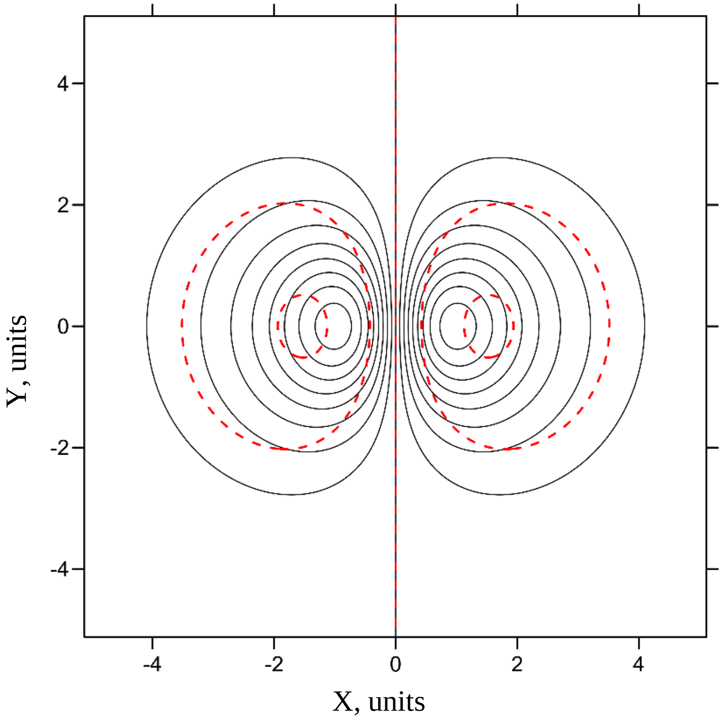
$$\psi_{h,x,y}^z(\xi,\eta)=\frac{1}{h}\psi_0\left(\frac{\xi-x}{h},\frac{\eta-y}{h}\right)=\frac{1}{h}\frac{2-(\frac{\xi-x}{h})^2-(\frac{\eta-y}{h})^2}{\left(\sqrt{(\frac{\xi-x}{h})^2+(\frac{\eta-y}{h})^2+1}\right)^5}=\underline{\frac{1}{h}h^3\frac{2h^2-(\xi-x)^2-(\eta-y)^2}{\left(\sqrt{(\xi-x)^2+(\eta-y)^2+h^2}\right)^5}}.$$

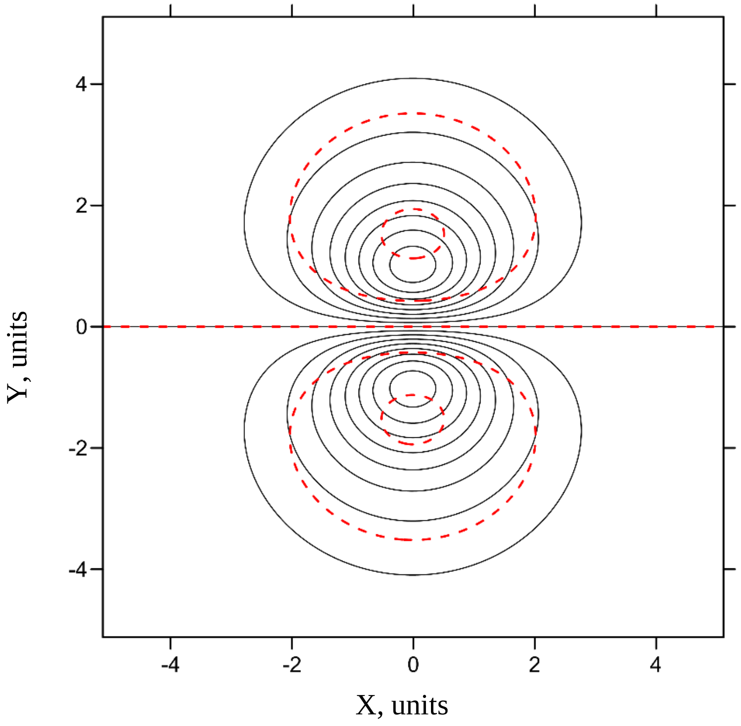
Poisson wavelets 1-st order

Similarly, we can construct Poisson wavelets that correspond to horizontal derivatives:

$$\psi^{x}_{h,x,y}(\xi,\eta)=\frac{1}{h}h^{3}\frac{3h(\xi-x)}{\left(\sqrt{(\xi-x)^{2}+(\eta-y)^{2}+h^{2}}\right)^{5}}$$

$$\psi^{y}_{h,x,y}(\xi,\eta)=\frac{1}{h}h^{3}\frac{3h(\eta-y)}{\left(\sqrt{(\xi-x)^{2}+(\eta-y)^{2}+h^{2}}\right)^{5}}.$$

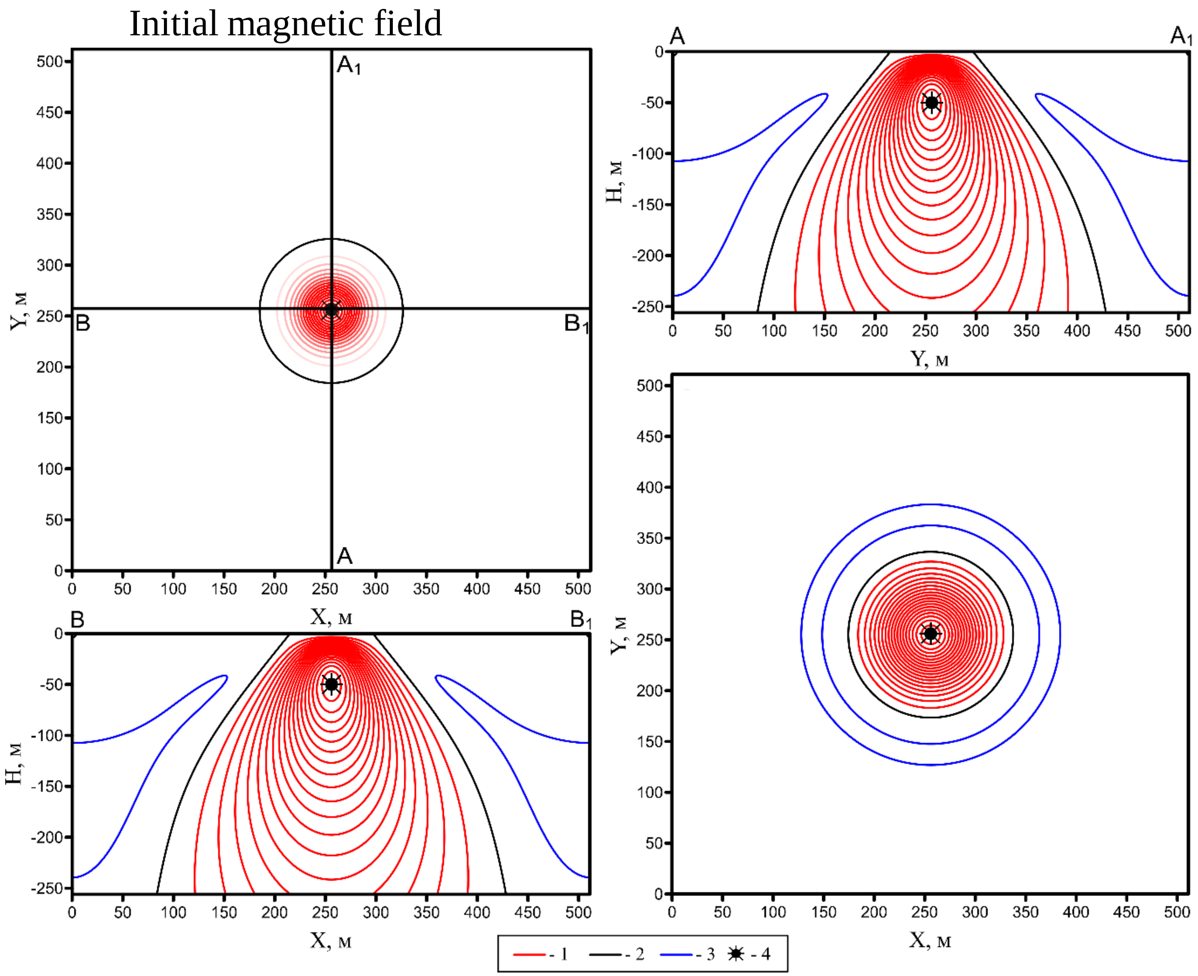




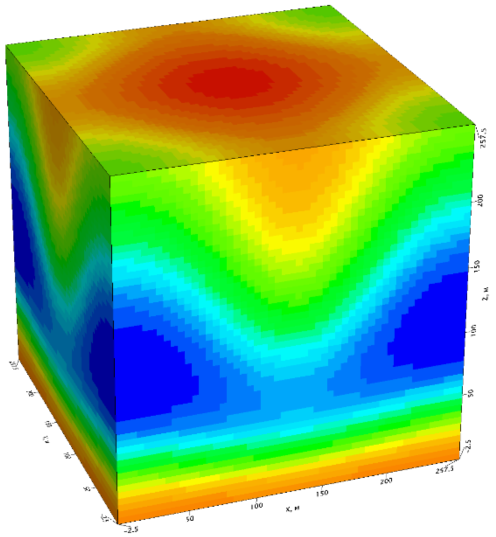
Solid line- h = 1, red dashed lines - h = 2.

Identification of Sources of Potential Fields

Wavelet-spectrum of magnetic dipole’s field



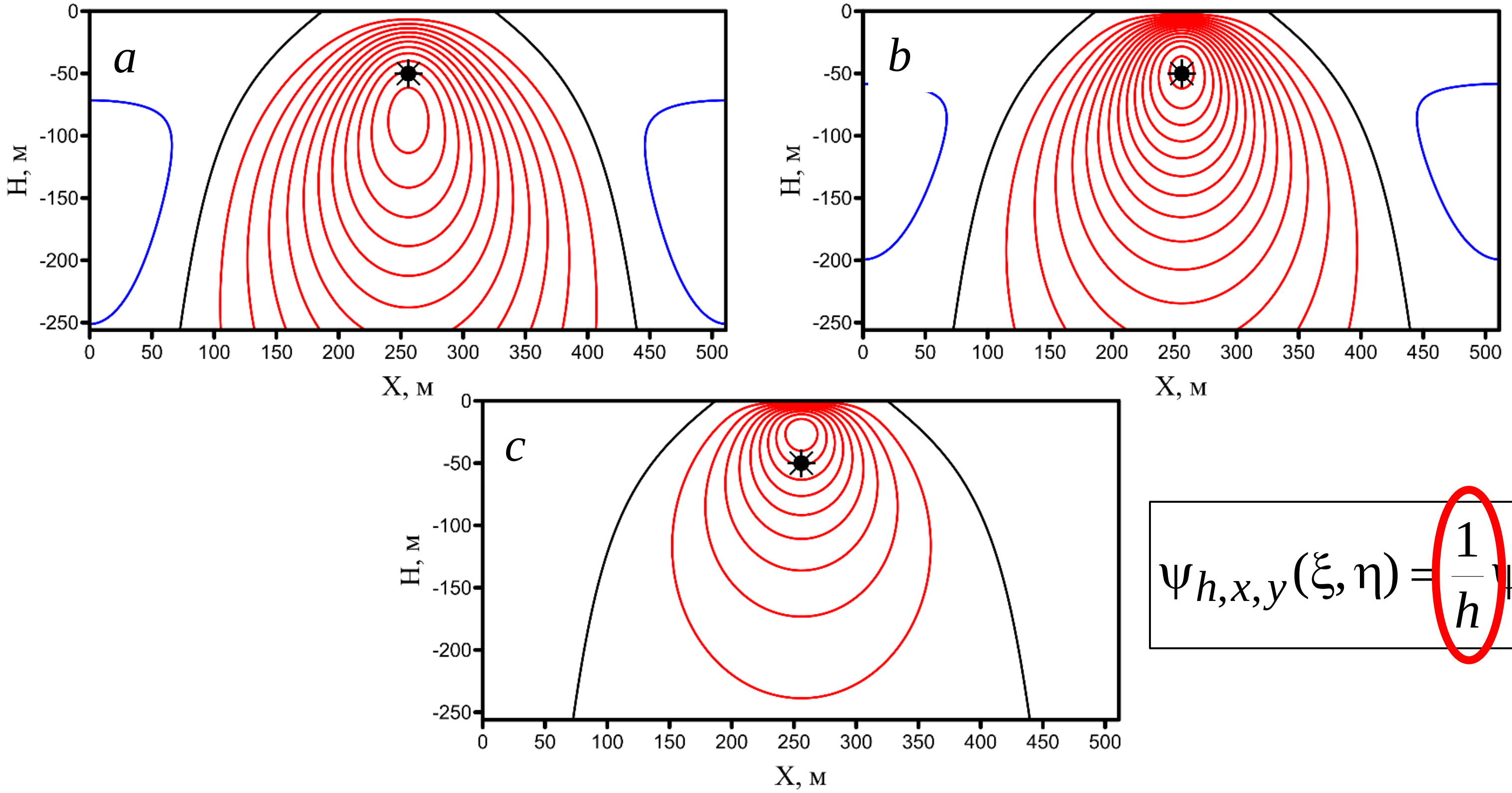
Slices of the wavelet-spectrum: 1 – positive isolines; 2-zero isolines; 3-negative isolines; 4-source position



Volume representation of the wavelet-spectrum

Identification of Sources of Potential Fields

Wavelet-spectrum of gravity field

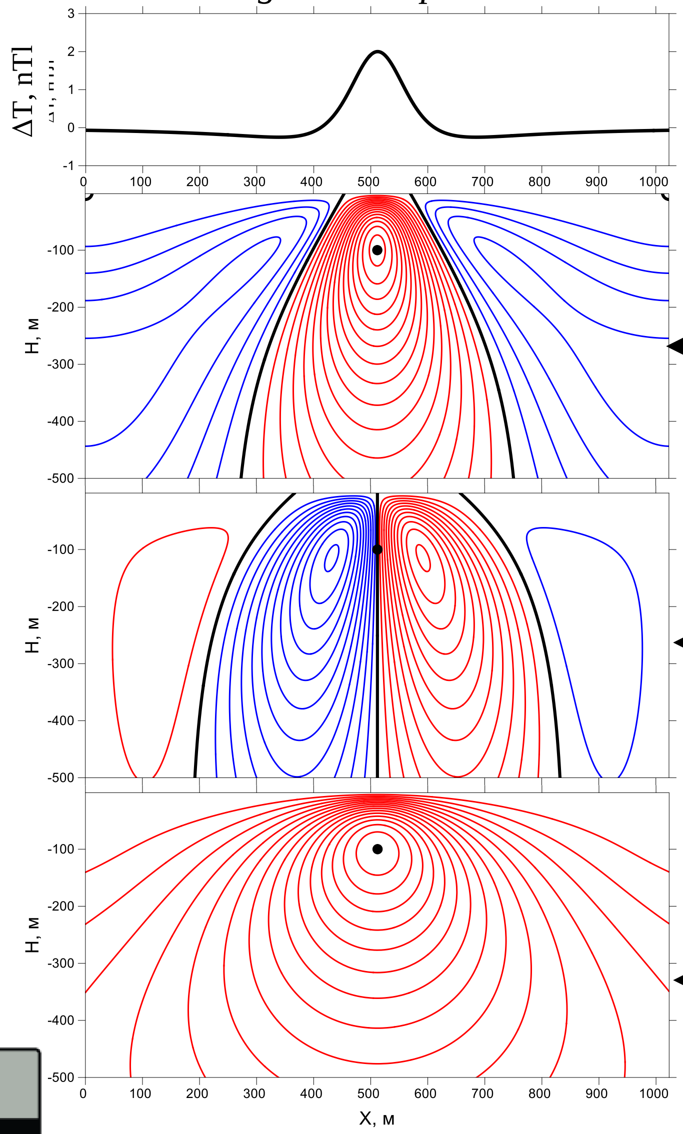


$$\psi_{h,x,y}(\xi,\eta)=\frac{1}{h}\psi_0\left(\frac{\xi-x}{h},\frac{\eta-y}{h}\right)$$

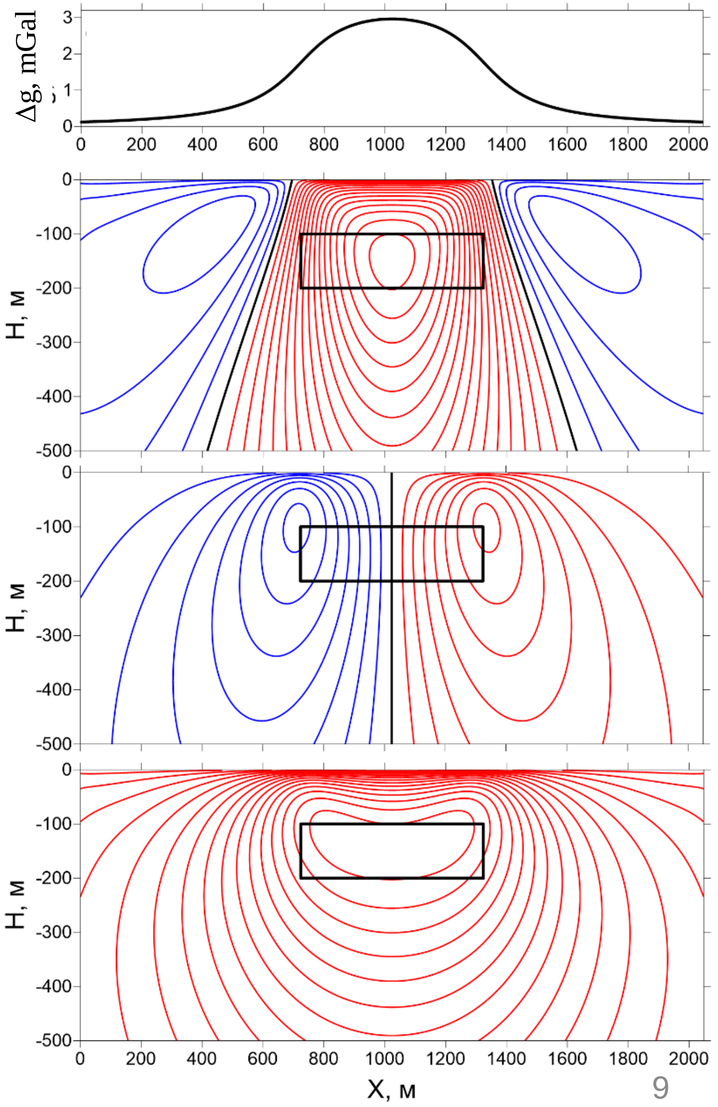
Slices of the wavelet-spectrum for the field V_z, the resulting wavelet-transform with a symmetric 1st-order Poisson wavelet with **different scale coefficients**: a – $1/h$, b – $1/h^{3/2}$, c – $1/h^2$.

Identification of Sources of Potential Fields by different types of Poisson wavelets

Magnetic field of vertically magnetized dipole



Gravity field of prism



1st order Poisson wavelet-spectrums:

W_z – corresponds to the **vertical** derivative

W_x – corresponds to the **horizontal** X-direction derivative

Module (amplitude) wavelet-spectrum:

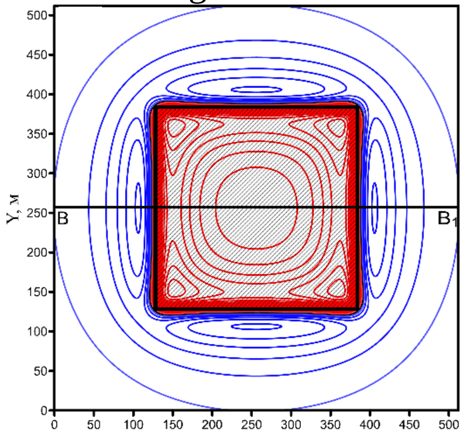
$$|W| = \sqrt{W_x^2 + W_y^2 + W_z^2}$$

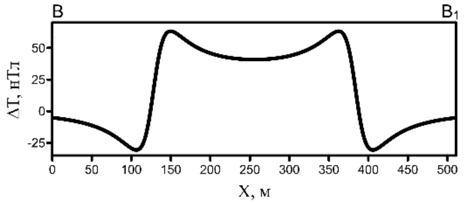
Identification of Sources of Potential Fields

Magnetic field's of the plate wavelet-spectrum calculated by 1st-order Poisson wavelets with a scale coefficient $1/h^2$

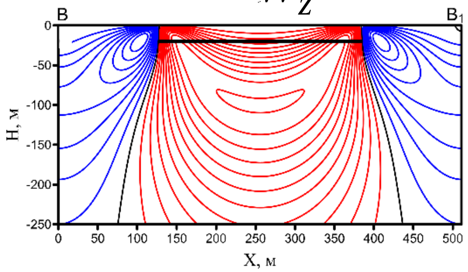
For localization of singular points it is possible to use Poisson wavelets of various types (correspond to different derivatives of the Poisson kernel), different orders and different scale coefficients.

Initial magnetic field

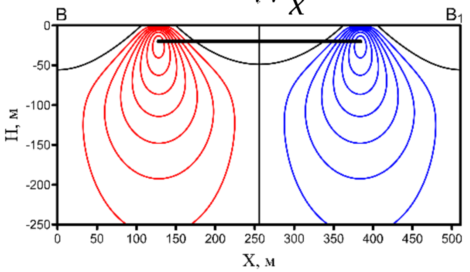




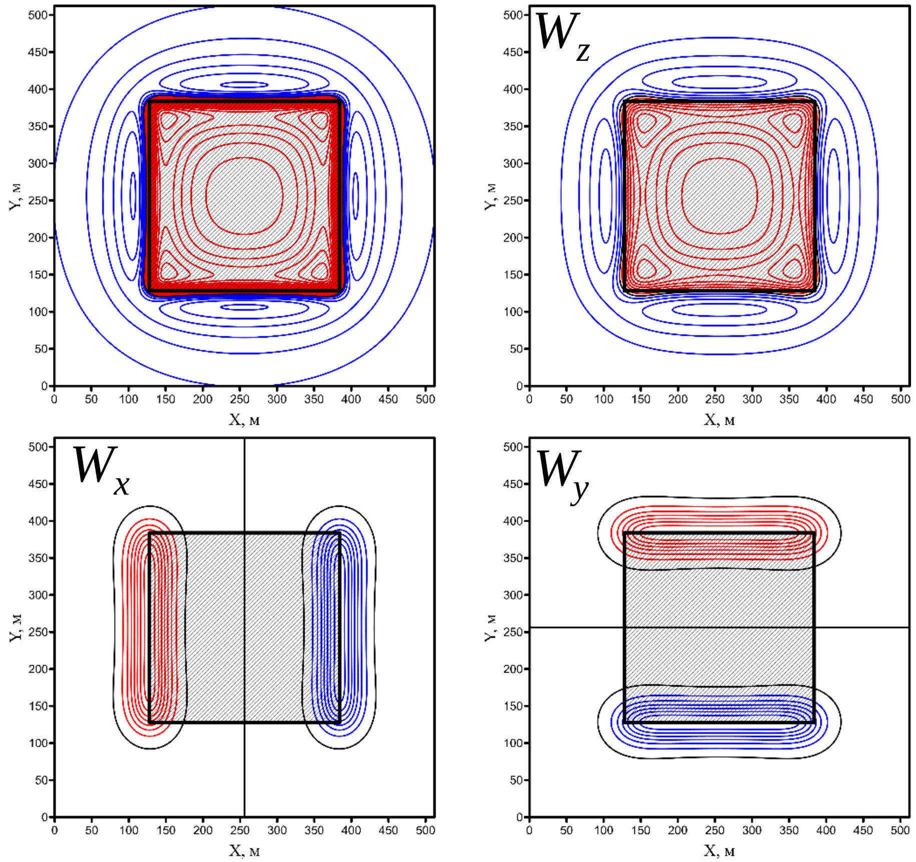
W_z



W_x



Horizontal slices of wavelet-spectrum at the source depth



Reverse wavelet-transform

If the source function$g(\xi,\eta)$ has mean value equal to zero, and its wavelet-spectrum was obtained by convolution with axisymmetric wavelets satisfy the conditions, we can perform two-dimensional inverse continuous wavelet-transform to function $W(h,x,y)$ to recover the original signal $g(\xi,\eta)$:

$$g(x,y)=\frac{1}{C_\psi}\int_0^\infty \frac{1}{h^3}\left(\frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(h,x,y)\psi_0\left(\frac{\xi-x}{h},\frac{\eta-y}{h}\right)dx\,dy\right)dh$$

C_ψ – constant defined by the function ψ_0:

$$C_\psi=\int_0^\infty\int_0^\infty \frac{\left|\hat{\psi}_0(\omega_x,\omega_y)\right|^2}{\omega}d\omega_x d\omega_y$$

$\hat{\psi}_0(\omega)$ – Fourier transformation of function $\psi_0(x,y)$, $\omega=\sqrt{\omega_x^2+\omega_y^2}$

Analytical solution of integral:

$$C_\psi=\pi^2\frac{1}{2}\frac{2N-1}{2}\frac{2N-2}{2}\cdots\frac{1}{2}=\pi^2\frac{(2N-1)!}{2^{2N}}$$ where N – order of Poisson wavelet

Reverse wavelet-transform

Consider the reverse wavelet-transform of the wavelet-spectrum calculated by a 1st-order Poisson wavelet corresponding to the vertical derivative:

$$\hat{g}(\omega_x,\omega_y)=\frac{1}{C_\psi}\int_0^h \frac{1}{h^3}\frac{1}{h}\hat{W}(h,\omega_x,\omega_y)h^3(\pi|\omega|e^{-|\omega|h})dh=$$

$$=\frac{1}{C_\psi}\int_0^h \frac{1}{h^3}\frac{1}{h}\left(\frac{1}{h}\hat{g}(\omega_x,\omega_y)h^3(\pi|\omega|e^{-|\omega|h})\right)h^3(\pi|\omega|e^{-|\omega|h})dh=$$

$$=\frac{1}{C_\psi}|\omega|^2\pi^2\hat{g}(\omega_x,\omega_y)\int_0^h h(e^{-2|\omega|h})dh=\frac{1}{C_\psi}|\omega|^2\pi^2\hat{g}(\omega_x,\omega_y)\frac{-1}{2|\omega|}\int_0^h hde^{-2|\omega|h}=$$

$$=\frac{1}{C_\psi}|\omega|^2\pi^2\hat{g}(\omega_x,\omega_y)\frac{-1}{2|\omega|}(he^{-2|\omega|h}\Big|_0^\infty)\int_0^h e^{-2|\omega|h}dh=$$

$$=\frac{1}{C_\psi}|\omega|^2\pi^2\hat{g}(\omega_x,\omega_y)\frac{-1}{2|\omega|}(he^{-2|\omega|h}\Big|_0^\infty)\frac{-1}{2|\omega|}e^{-2|\omega|h}\Big|_0^\infty=\boxed{\frac{1}{C_\psi}\hat{g}(\omega_x,\omega_y)\frac{\pi^2}{4}}$$

Reverse wavelet-transform

Consider the reverse wavelet-transform of the wavelet-spectrum calculated by a 1st-order Poisson wavelet corresponding to the horizontal X-direction derivative:

$$\hat{g}(\omega_x,\omega_y)=\frac{1}{C_\psi}\int_0^h \frac{1}{h^3}\frac{1}{h}\hat{W}(h,\omega_x,\omega_y)h^3(\pi\omega_x e^{-|\omega|h})dh=$$

$$=\frac{1}{C_\psi}\int_0^h \frac{1}{h^3}\frac{1}{h}\left(\frac{1}{h}\hat{g}(\omega_x,\omega_y)h^3(\pi\omega_x e^{-|\omega|h})\right)h^3(\pi\omega_x e^{-|\omega|h})dh=$$

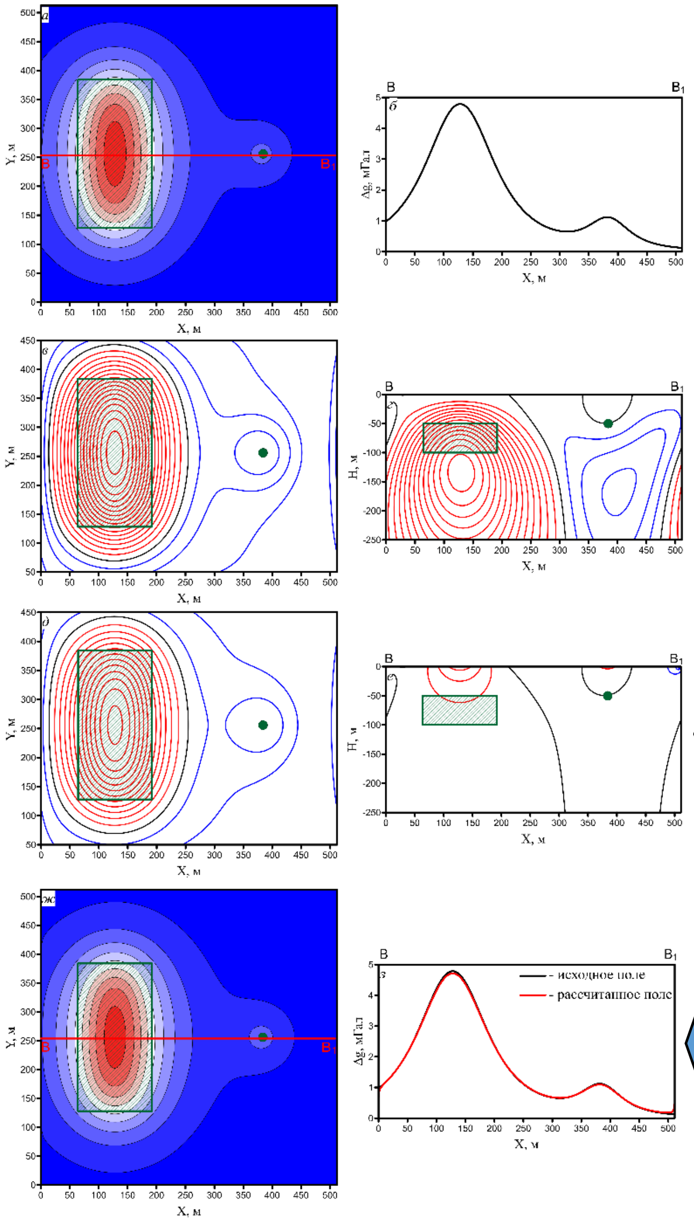
$$=\frac{1}{C_\psi}\omega_x{}^2\pi^2\hat{g}(\omega_x,\omega_y)\int_0^h h(e^{-2|\omega|h})dh=\frac{1}{C_\psi}\omega_x{}^2\pi^2\hat{g}(\omega_x,\omega_y)\frac{-1}{2|\omega|}\int_0^h hde^{-2|\omega|h}=$$

$$=\frac{1}{C_\psi}\omega_x{}^2\pi^2\hat{g}(\omega_x,\omega_y)\frac{-1}{2|\omega|}(he^{-2|\omega|h}\Big|_0^\infty)\int_0^h e^{-2|\omega|h}dh=$$

$$=\frac{1}{C_\psi}\omega_x{}^2\pi^2\hat{g}(\omega_x,\omega_y)\frac{-1}{2|\omega|}(he^{-2|\omega|h}\Big|_0^\infty)\frac{-1}{2|\omega|}e^{-2|\omega|h}\Bigg|_0^\infty=\frac{1}{C_\psi}\hat{g}(\omega_x,\omega_y)\frac{\pi^2}{4}\frac{\omega_x{}^2}{|\omega|^2}$$

It is impossible to restore the original function from the wavelet spectrum of asymmetric Poisson wavelets.

Wavelet-spectrum conversion to equal density and magnetization distribution



Using wavelet-spectrum calculated by a 1st-order Poisson wavelet, it is possible to reconstruct equal density distribution that creates the original gravity field.

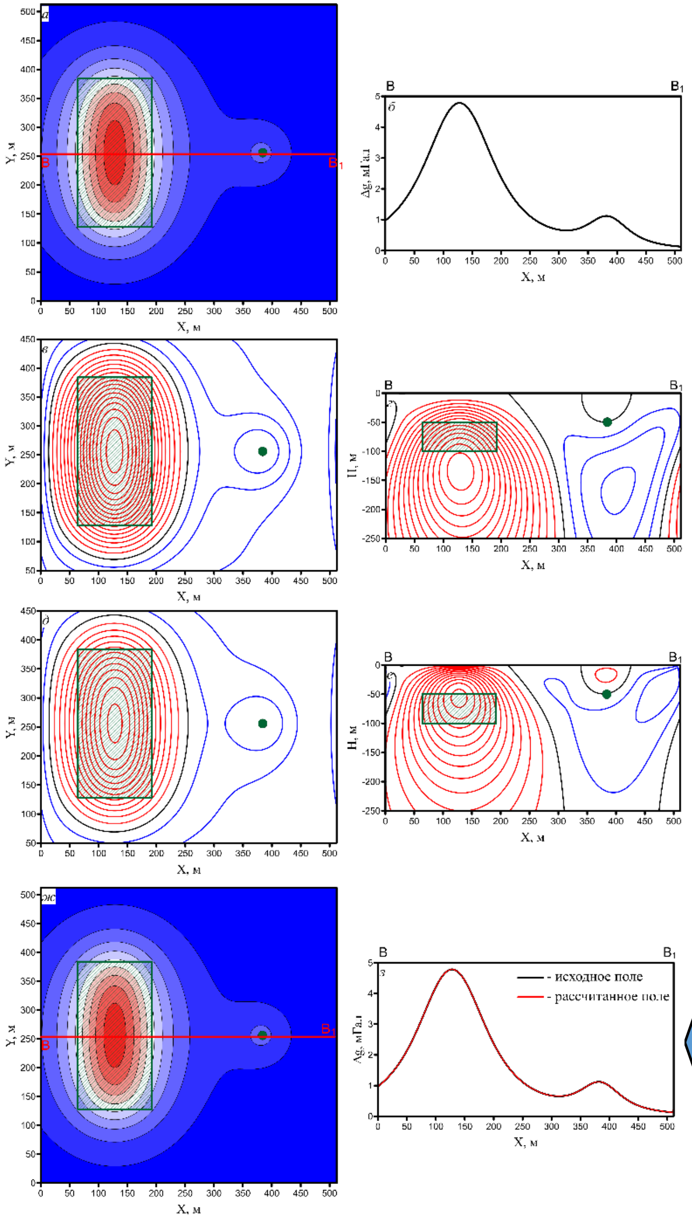
Wavelet-spectrum calculated by a 1st-order Poisson wavelet corresponding to the vertical derivative

$$\delta(h,x,y)=\frac{1}{\pi G}V_{zz}(h,x,y)=\frac{1}{2\pi}\frac{h}{h^3}\frac{1}{\pi G}W(h,x,y)=$$

$$=\frac{1}{2\pi^2 h^2 G}W(h,x,y)$$

Gravity effect of the density cube.

Wavelet-spectrum conversion to equal density and magnetization distribution



Using wavelet-spectrum calculated by a 1st-order Poisson wavelet, it is possible to reconstruct the equal density (or magnetization) distribution that creates the original field V_{zz}.

Wavelet-spectrum calculated by a 1st-order Poisson wavelet corresponding to the vertical derivative.

$$\delta(h,\xi,\eta)=\frac{1}{GC_{\psi}h}W(h,x,y)\ \Bigg|\ \left(I(h,\xi,\eta)=\frac{1}{C_{\psi}h}W(h,x,y)\right)$$

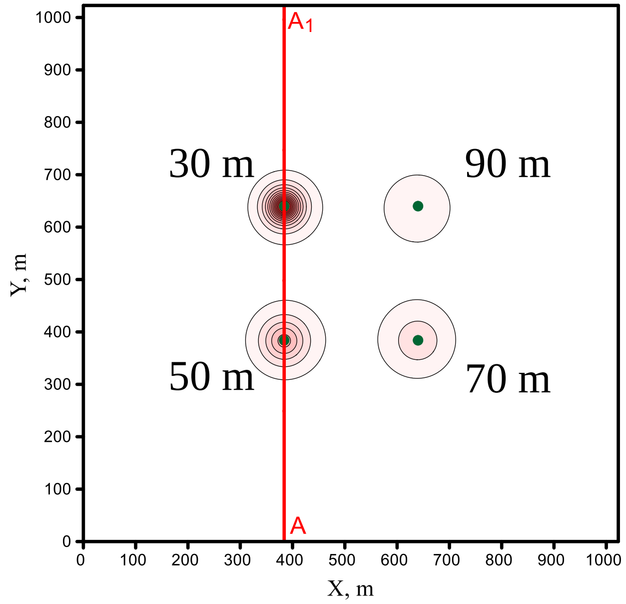
Density

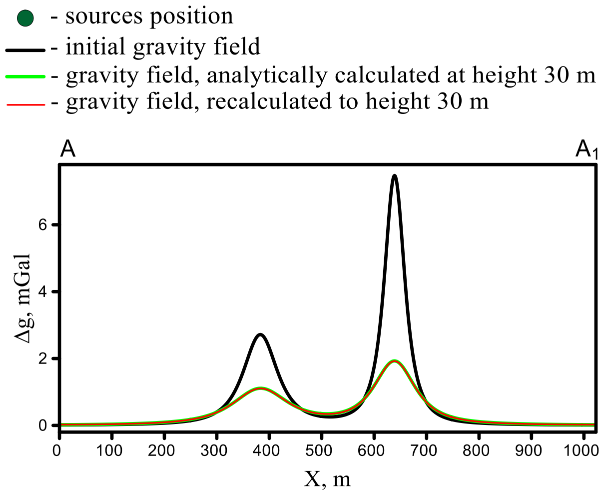
Magnetization in the SGS system

V_{zz} field of the density cube.

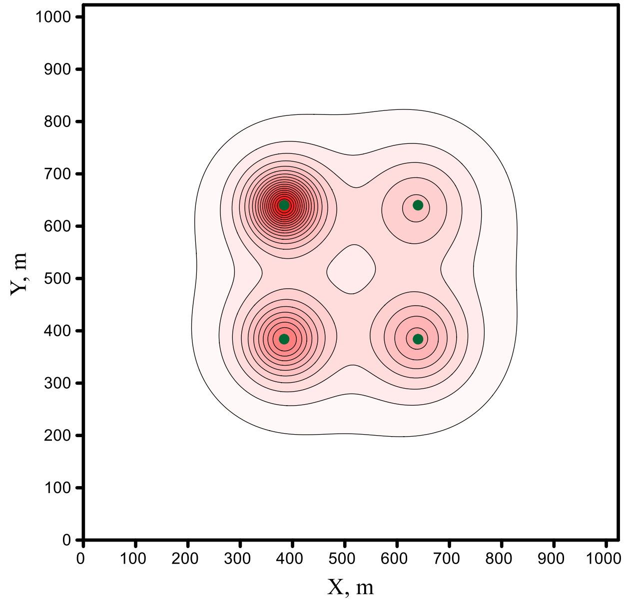
Upward continuation algorithm based on Poisson wavelets

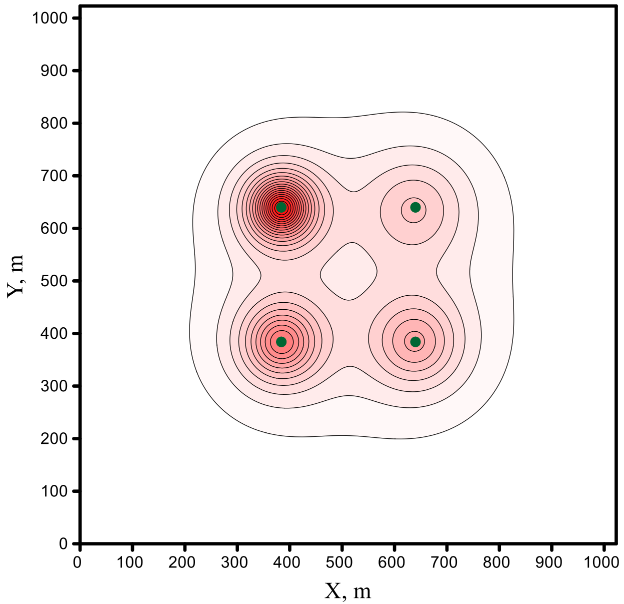
If we know equal density distribution (magnetization), it is possible to build algorithm for calculating derivatives and upward continuation by direct effect calculation.





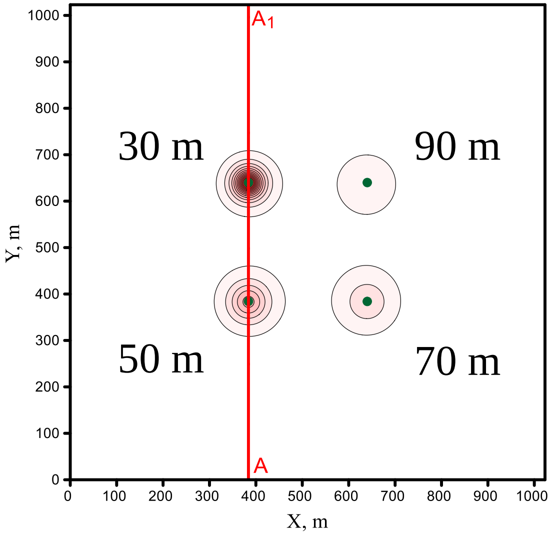
Gravity field analytically calculated at the level of 30 m

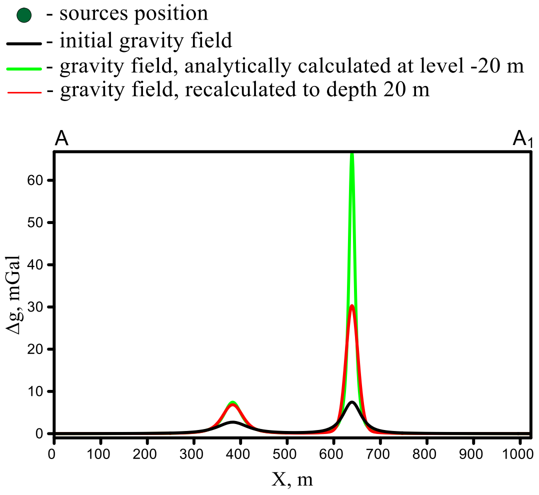


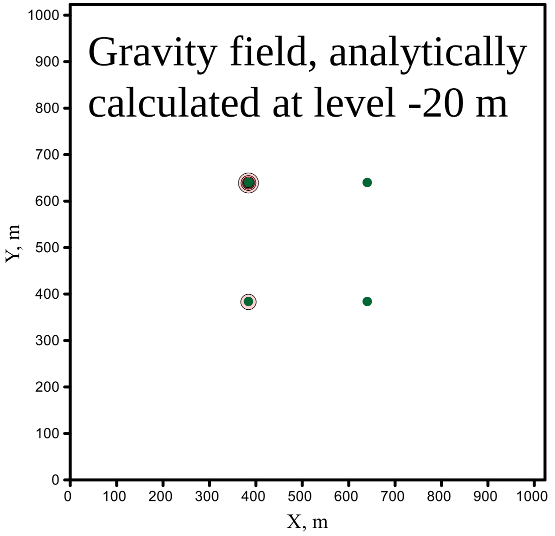


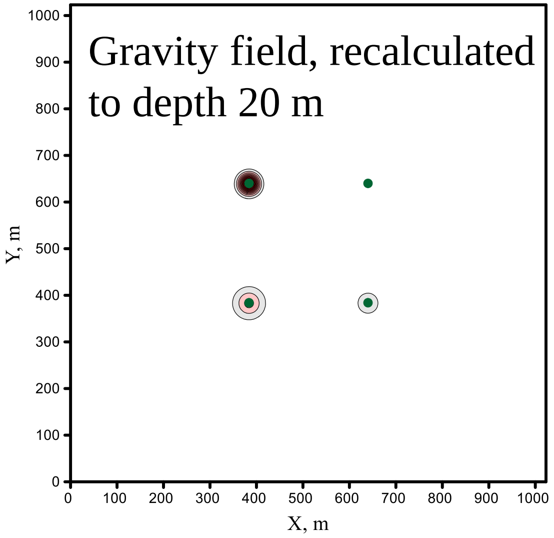
Gravity field recalculated to level of 30 m

Downward continuation algorithm based on Poisson wavelets

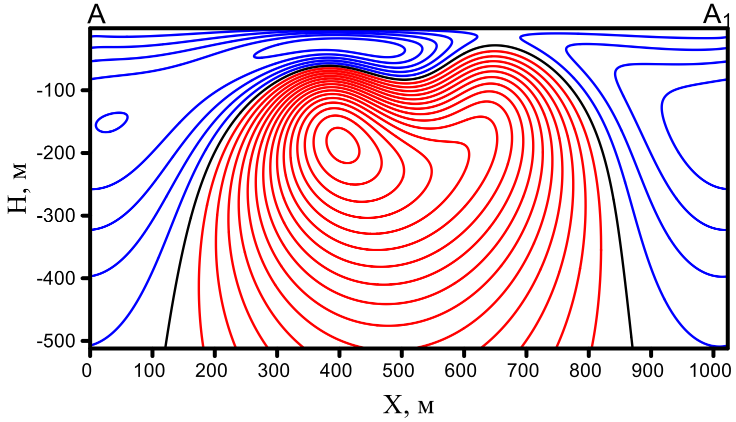






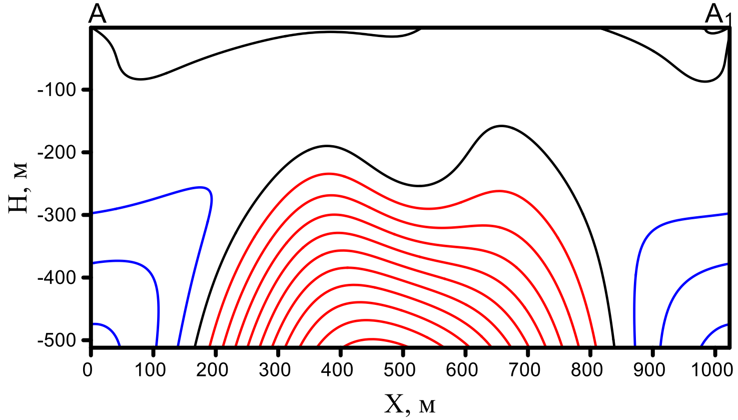


Cross section of densities calculated by a 1st-order Poisson wavelet



1. For downward continuation we need to delete sources that are above the continuation level/

A cross section of densities calculated by a 3rd-order Poisson wavelet

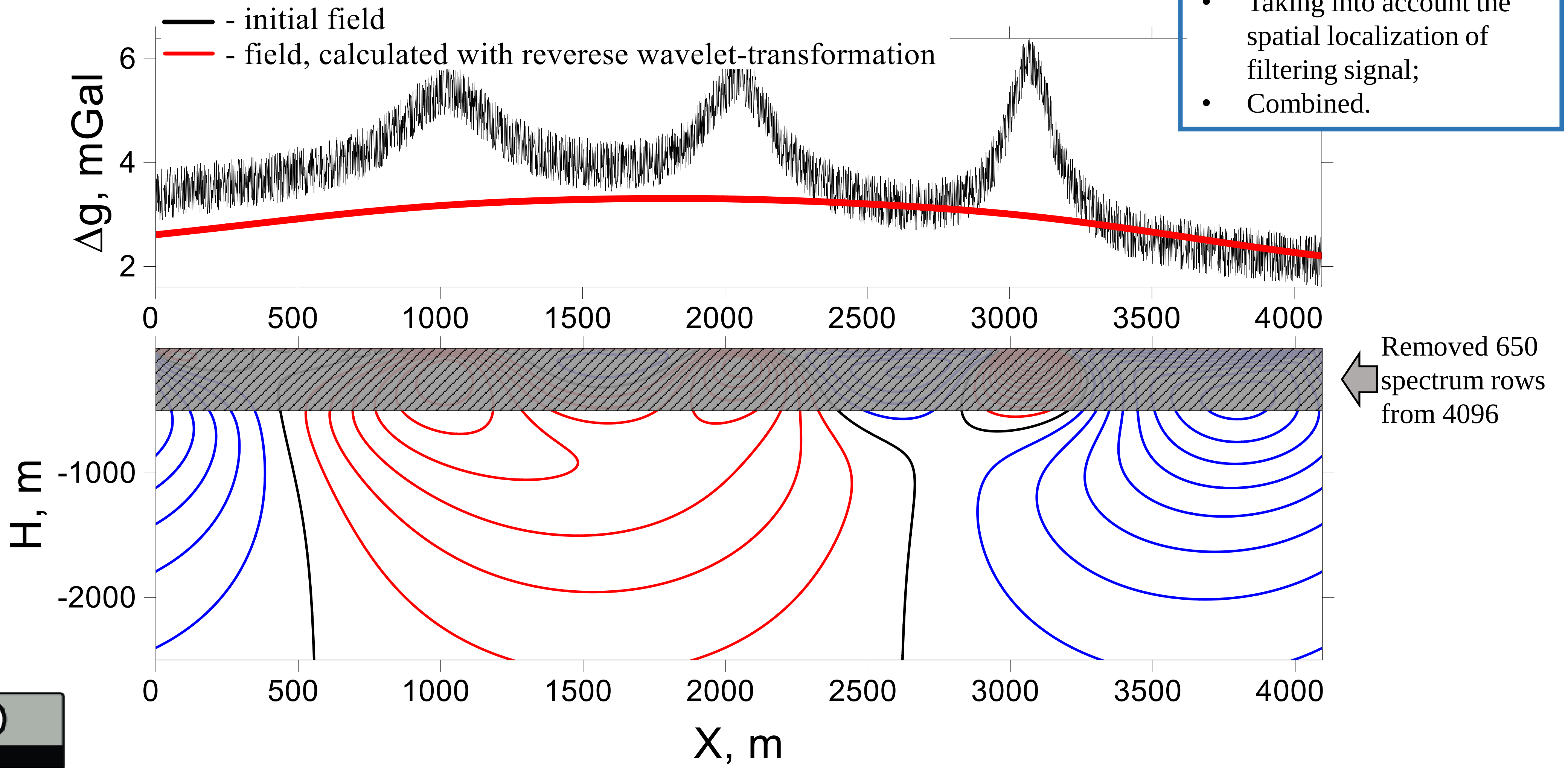


2. To minimize influence of remoted rows it is better to use Poisson wavelets of higher orders

Gravity and magnetic fields filtering

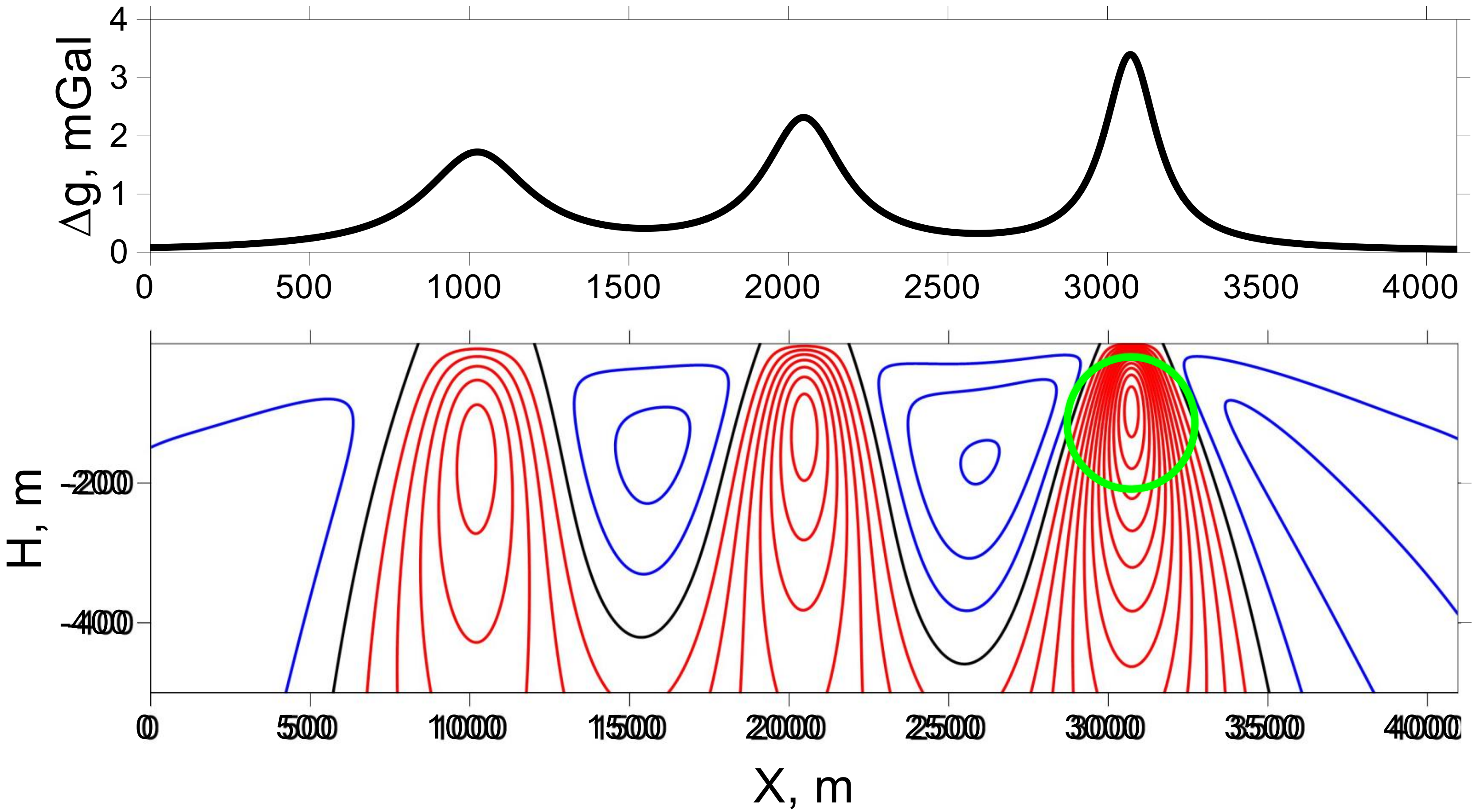
Initial signal → Wavelet-spectrum's calculation → Filtering of the wavelet-spectrum → Reverse wavelet-transformation →

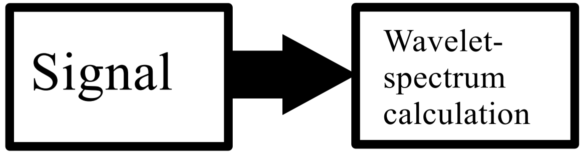
Output signals:
- Separated by levels of wavelet-spectrum (frequency);
- Taking into account the spatial localization of filtering signal;
- Combined.



Filtering with t king into account the spatial localization of filtering signal.

Step 1



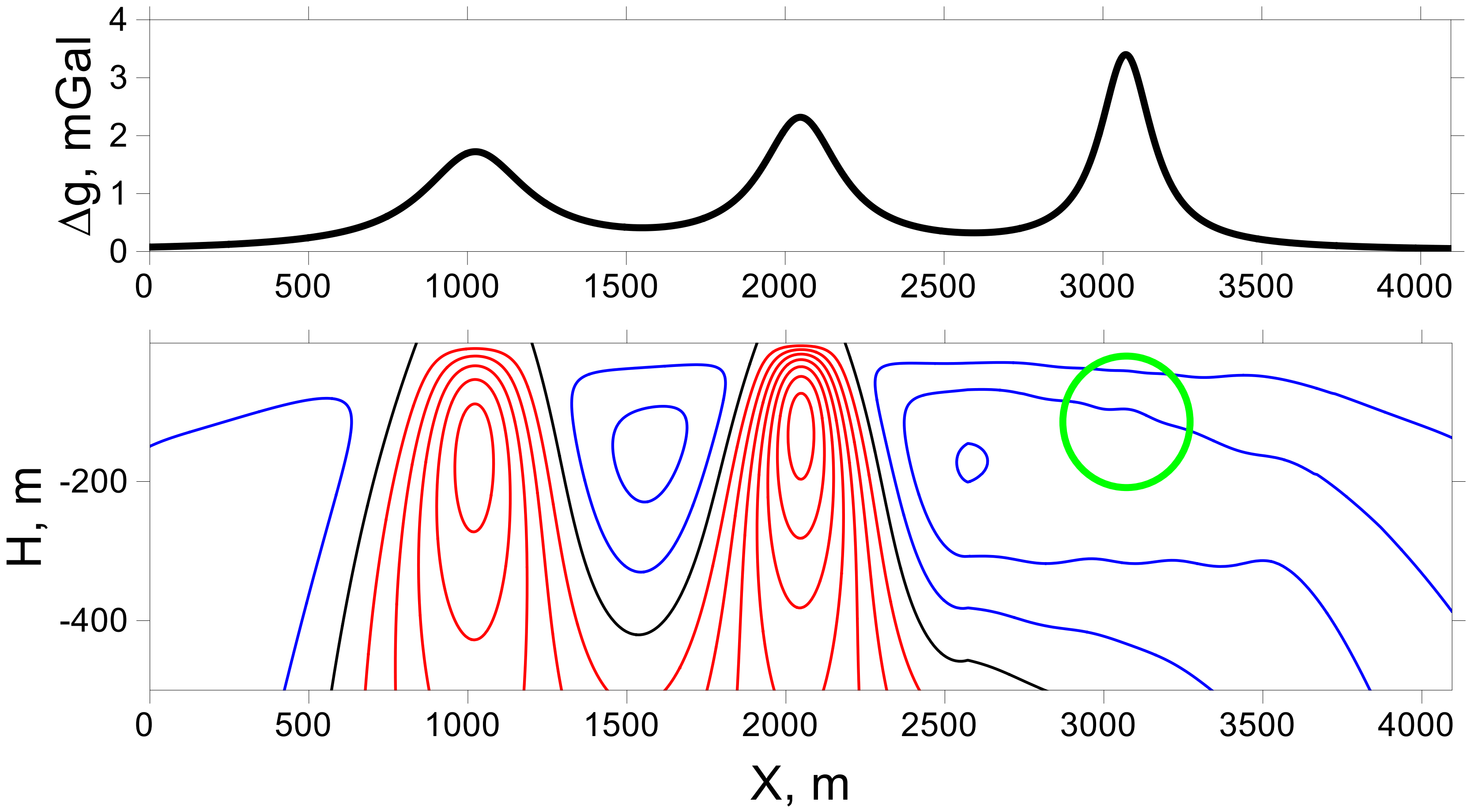


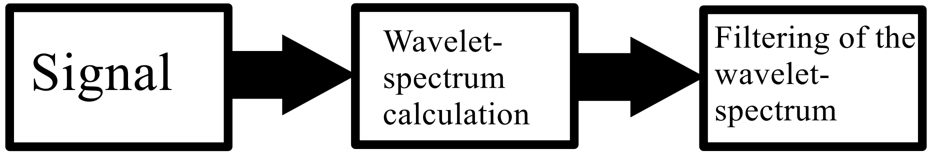
- initial field
- field, calculated with reverese wavelet-transformation
- area of wavelet-spectrum filtered extremum

Gravity and magnetic fields filtering

Filtering with t king into account the spatial localization of filtering signal.

Step 2



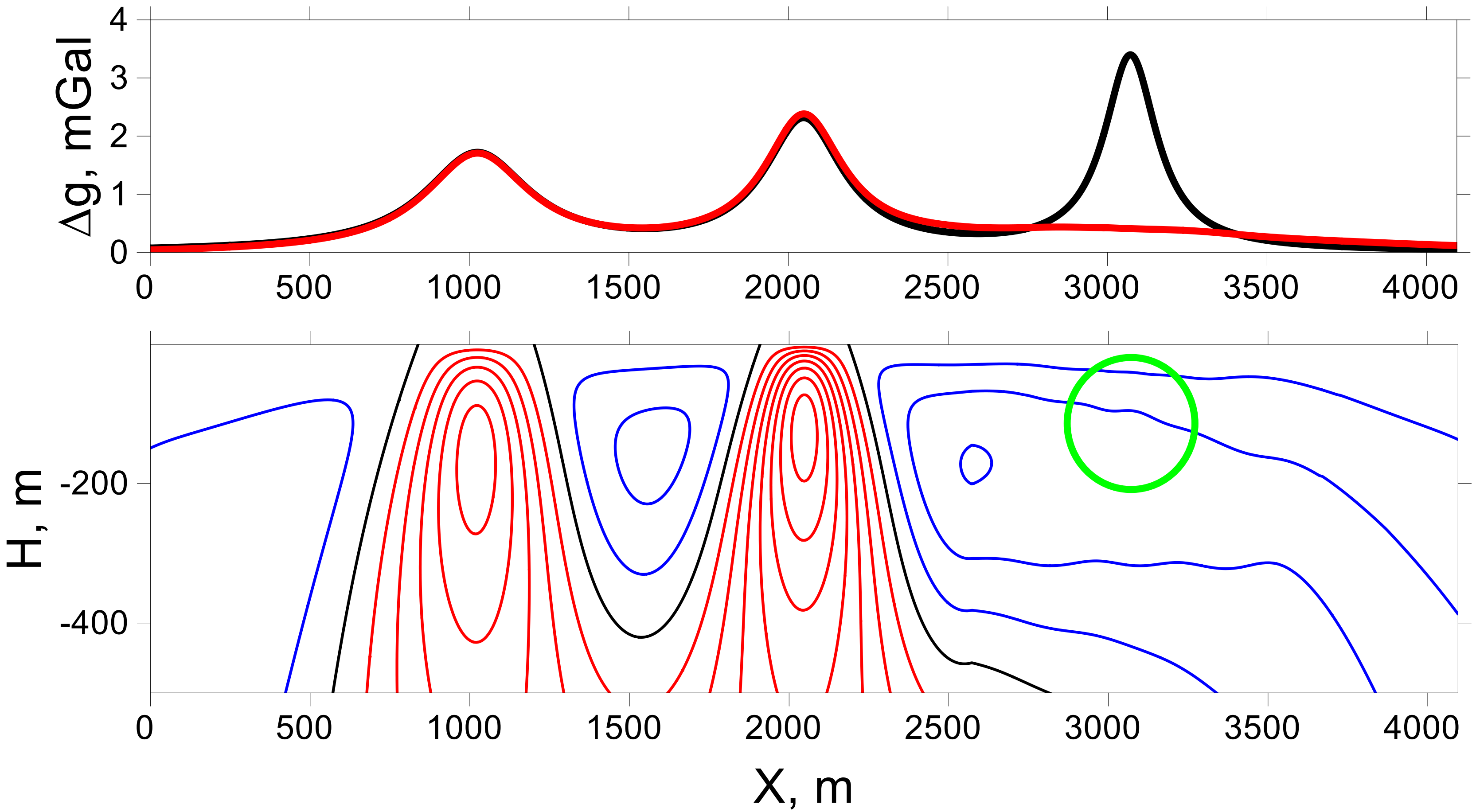


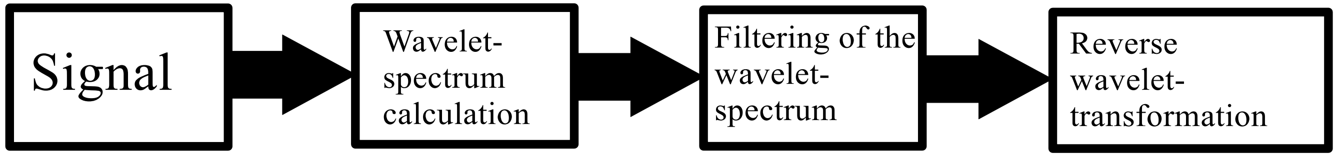
- initial field
- field, calculated with reverese wavelet-transformation
- area of wavelet-spectrum filtered extremum

Gravity and magnetic fields filtering

Filtering with t king into account the spatial localization of filtering signal.

Step 3





- initial field
- field, calculated with reverese wavelet-transformation
- area of wavelet-spectrum filtered extremum

Conclusion

1. It is shown that when analyzing the wavelet-spectrum calculated by wavelets based on the Poisson kernel, it is possible to distinguish the position of singular points of gravity and magnetic fields.

2. Wavelet-transforms by a group of Poisson wavelets can be applied to solve the following problems:
   - correct (conversion to upper half-space, conversion to equal density or magnetization distribution);
   - incorrect (downward continuation);
   - filtering fields with consideration for spatial localization of anomalies (filtering of wavelet-spectrum).

THANKS FOR YOUR ATTENTION!