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The Laplace plane of Mercury

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1. Introduction

In a perfect system, the **Laplace plane** (LP) is the plane about which the orbital inclination remains constant throughout the precessional cycle. It can be viewed as the “mean orbital plane”. For Mercury, knowing this plane is important because the Cassini state and the equilibrium obliquity refer to it.
We define a general Laplace plane based on geometrical considerations. The Laplace plane for a regular motion is deduced from the previous definition by adding dynamical constraints of constant inclination and regular precession around a fixed axis. A comparison to the simplified model of the secular potential is given.
Then we apply it to the true orbit of Mercury and compare different LP estimations.

2. Definition of the angles

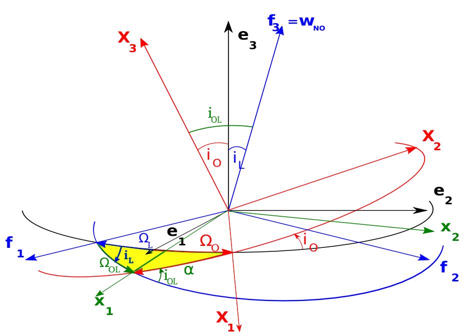

Figure 1: Positions of the Laplace plane ($\mathbf{f_1 f_2}$ in blue) and orbital planes (in red) and the different angles in the problem. The ecliptic frame $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is in black. The orbital plane is precessing around the Laplace plane.

From trigonometric equations in the yellow spherical triangle, we can find the expression for the orbital elements (i_O and Ω_O) as a function of the angles related to the Laplace plane for a synthetic case.
The α angle is the $\mathbf{X_1 x_1}$ angle from the ascending node of the orbit on the inertial plane to the ascending node of the orbit on the Laplace plane.

i_{OL} is the angle between the vector $\mathbf{w}$ (or the normalized vector $\mathbf{w}_{NO}$) and the orbit normal $\mathbf{x}_3$. μ is the precession rate around the LP.

3. Geometrical definition of the LP

Using rotation matrix, the position of the instantaneous rotation vector (or orbital angular velocity vector) is:

$$\mathbf{w} = -\dot{\mathbf{M}}\mathbf{M}^{-1} = \big(\cos\Omega_\mathrm{O} i'_\mathrm{O} + \sin i_\mathrm{O} \sin\Omega_\mathrm{O}\,\alpha', \quad (1)$$
$$\sin\Omega_\mathrm{O} i'_\mathrm{O} - \sin i_\mathrm{O}\cos\Omega_\mathrm{O}\,\alpha', \Omega'_\mathrm{O} + \cos i_\mathrm{O}\,\alpha'\big)$$

Another way to find $\mathbf{w}$ is the vectorial product [3]:

$$\mathbf{w} \times \mathbf{x}_3 = \mathbf{v} \quad (2)$$

$\mathbf{v}$ is the orbital velocity.
In Eq. 1, the only unknown parameter is the time derivative of the angle $\alpha(t)$. Since we do not assume anything for the angles Ω_{OL}, i_{OL}, i_L and Ω_L in this section, the angle $\alpha(t)$ (and then the position of $\mathbf{x}_1$ and $\mathbf{x}_2$ with respect to $\mathbf{X}_1$ and $\mathbf{X}_2$) can be chosen freely. Choosing $\alpha(t)$ will fix the position of our generalized Laplace plane.
Since all the $\mathbf{w}$ vectors belong to the same plane as the orbit normal (plane perpendicular to $\mathbf{v}$), the orbital angular velocity vectors $\mathbf{w}$ given by eq. 1 for any value of the α' parameter give the same coplanarity condition for the Cassini state.

There is a degree of freedom because we describe the motion of a plane, not a body.

The normal to the Laplace plane is different from the angular momentum vector of the planet.

Quantities related to the vector w

By taking the norm of the vectorial product (2), we find a useful equation:

$$\mu \sin i_{OL} = \sqrt{\sin^2 i_\mathrm{O}\,\Omega'^2_\mathrm{O} + i'^2_\mathrm{O}} \quad (3)$$

This equation is independent of the choice of the parameter α' and is equal to the orbit velocity norm around the ecliptic pole.

4. The LP for a regular motion

For a perfectly regular motion, we impose 3 dynamical constraints:
- a fixed LP position ($di_L/dt = d\Omega_L/dt = 0$)
- a constant inclination ($d\cos i_{OL}/dt = 0$)
- a constant precession rate ($d\mu/dt = 0$)

Then the free parameter α' is given by:

$$\alpha'_{LP} = \cos i_O \Omega'_O + \sin i_O \Omega''_O / i'_O \quad (4)$$

5. Secular potential

For the natural satellites, the potential usually takes into account the planet oblateness, the external or internal perturbers, the Sun,...
For Mercury, the secular potential includes the secular effect of all the planets on its motion; it has no mutual planetary perturbations.

If the potential is simple and for small inclinations, the solution for the motion is the sum of a free part and a forced part. The secular Laplace plane position is the forced part and is given by the mean value of the orbital coordinates [4].

6. Application to Mercury orbit

Why is it tricky to compute the LP position for Mercury?
- because the orbit motion not perfectly regular (the deviation of the true motion of Mercury from a regular motion, given by Eq. 3, has variations of about 1% over 2 ky).
- the orbit is known over a short interval (about 20 ky $\ll$ precession period of the orbit $\sim$ 300 ky).
- numerical computation of the orbital elements and their time derivatives (averaged on which timescale?).

How to fix the free degree of freedom numerically (Fig. 2)?
- Fit a cone to the ephemerides [3]
- Arbitrarily choose the free parameter ($\alpha' = 0$ [1])
- 5 parameters fit (Ω_{OL}, i_{OL}, i_L, Ω_L and μ)

7. The Cassini state

The conditions defining the Cassini state:
The spin axis, orbit normal, and normal to the Laplace plane are coplanar while the obliquity remains constant.

The equilibrium obliquity of Mercury (3:2 resonance) in the Cassini state links the LP parameters (i_{OL} and μ) to the interior properties (moment of inertia C).

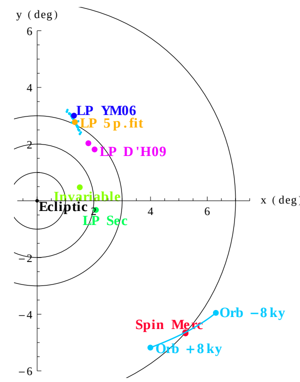

Figure 2: Projection in the ecliptic plane

Uncertainty on C/MR^2 due to the uncertainty on the LP from different authors ([1], [3]) $\approx 0.1\%$

Target uncertainty for C/MR^2 from the ESA Bepi-Colombo spacecraft $\simeq 0.3\%$ [2]

8. Laplace plane: Summary

- For a regularly precessing orbit, the Laplace plane position can be found using a geometrical approach and dynamical constraints.
- If the motion is not perfect, the generalized Laplace plane is moving with time or the orbital inclination is not constant with time. Its definition is less clear.
- For Mercury, the orbital motion is known on a very short interval. It is difficult to evaluate the LP position precisely.
- However using the equation for the equilibrium obliquity of the Cassini state, the uncertainty on the moment of inertia due to the uncertainty on the LP parameters is about 0.1%.

References

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