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# Thermal-Gravitational Wind Equation for the Wind-Induced Gravitational Signature of Jupiter and Saturn

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## Abstract

The thermal wind equation has been used to calculate the external gravitational signature produced by zonal winds in the interiors of Jupiter and Saturn. We show however that in this application the thermal wind equation needs to be generalized to account for an associated gravitational perturbation. We refer to the generalized equation as the thermal-gravitational wind equation. The generalized equation represents a two-dimensional kernel integral equation with the Green's function in its integrand and is hence much more difficult to solve than the standard thermal wind equation. We develop an extended spectral method for solving the thermal-gravitational wind equation in spherical geometry. We then apply the method to a generic gaseous Jupiter-like object with idealized zonal winds. We demonstrate that solutions of the thermal-gravitational wind equation are substantially different from those of the standard thermal wind equation. We conclude that the thermal-gravitational wind equation must be used to estimate the gravitational signature of deep zonal winds in giant gaseous planets.

## 1. Introduction

Two different approaches have been adopted to compute the wind-induced density anomaly in the deep interiors of Jupiter, Saturn and the corresponding external gravitational signatures. The computed gravitational signatures, in turn, will be used to interpret the high-precision measurements of the external gravitational fields of Jupiter and Saturn to be carried out by the Juno and Cassini spacecrafts. The first approach is based on *the thermal wind equation* (the TWE) [2, 3, 4], a diagnostic relation given by

$$\rho' = C(r) + \frac{2r\Omega}{g_{static}(r)} \int_{\pi/2}^{\theta} \Big[ \cos\tilde{\theta} \frac{\partial}{\partial r} (\rho_{static} U) - \frac{\sin\tilde{\theta}}{r} \frac{\partial}{\partial \tilde{\theta}} (\rho_{static} U) \Big] \, \mathrm{d}\tilde{\theta}, \quad (1)$$

where $U(r,\theta)$ denotes the zonal winds, $\rho'(r,\theta)$ is the wind-induced density perturbation, $\boldsymbol{g}_{static} = \hat{\mathbf{r}} g_{static}(r)$ and $C(r)$ denotes an arbitrary function of $r$. The second approach makes the barotropic assumption – the density in fully compressible gaseous planets is a function only of the pressure. The barotropic model was adopted to study the effect of deep zonal winds on Jupiter's gravitational harmonics in spherical geometry [1] and in non-spherical geometry [5, 6].

We show, via both mathematical analysis and the numerical computation of simple models, that the TWE given by (1) is, in general, not valid for determining the density perturbation $\rho'(r,\theta)$ induced by deep zonal winds in giant gaseous planets such as Jupiter and Saturn. We point out that zonal flow produces not only the density perturbation $\rho'$ but also a concomitant gravitational perturbation $\boldsymbol{g}'(r,\theta)$ to the hydrostatic gravitational force $\boldsymbol{g}_{static}$. In terms of the mathematical formulation, an extra term representing the gravitational perturbation $\boldsymbol{g}'$ produced by the interior density perturbation $\rho'$ is of the same order of magnitude and, hence, must be retained. We then show that retaining the gravitational perturbation $\boldsymbol{g}'$ leads to *the thermal-gravitational wind equation* (TGWE), a two-dimensional kernel integral equation which, in contrast to the TWE (1), is much more difficult to solve. Through an analytical model for $\rho_{static}(r)$ in spherical geometry, the results of our calculation demonstrate that solutions of the TGWE are substantially different from those of the TWE and that the TWE, in general, cannot provide a reasonable approximation to the TGWE.

## 2. Derivation of the TGWE

Upon assuming that giant gaseous planets with mass $M$ are isolated and rotating about the $z$-axis with angular velocity $\Omega\hat{z}$, the governing equations in the rotating frame of reference are

$$2\Omega\hat{\boldsymbol{z}}\times\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p+\boldsymbol{g}+\frac{\Omega^2}{2}\boldsymbol{\nabla}\left|\hat{z}\times\boldsymbol{r}\right|^2, \quad (2)$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{u}\rho) = 0, \quad (3)$$

where $\boldsymbol{u}(\boldsymbol{r})$ represents the velocity of the zonal winds, $\boldsymbol{r}$ denotes the position vector with the origin at the center of figure, $p(\boldsymbol{r})$ is the pressure and $\rho(\boldsymbol{r})$ is the density. Suppose that the speed of the zonal flow $\boldsymbol{u}$ is small compared to the rotation speed of the planet, equations (2)–(3) can be then solved by making use of the expansions

$$p = p_{static}(r,\theta)+p'(r,\theta),$$
$$\rho = \rho_{static}(r,\theta)+\rho'(r,\theta),$$
$$\boldsymbol{g} = \boldsymbol{g}_{static}(r,\theta)+\boldsymbol{g}'(r,\theta),$$

where the leading-order solution, ($p_{static}, \rho_{static}$ and $\boldsymbol{g}_{static}$), represents the hydrostatic state of the rotating gaseous planet while $(p', \rho', \boldsymbol{g}')$ denotes the perturbations arising from the effect of the zonal winds $\boldsymbol{u}$.

The second-order problem, which describes the density anomaly $\rho'$ induced by the deep zonal flow $\boldsymbol{u}$ and the concomitant gravitational perturbation $\boldsymbol{g}'$ directly produced by $\rho'$, is governed by the equations

$$2\rho_{static}\left(\Omega\hat{z}\times\boldsymbol{u}\right) = -\boldsymbol{\nabla}p'+\boldsymbol{g}_{static}\rho'+\boldsymbol{g}'\rho_{static} \quad (4)$$
$$0 = \boldsymbol{\nabla}\cdot(\boldsymbol{u}\rho_{static}). \quad (5)$$

In deriving (4) – (5), we have neglected the small high-order terms which are of $\mathrm{O}(|\boldsymbol{g}'\rho'|)$ and $\mathrm{O}(|\boldsymbol{u}\rho'\Omega|)$ and we have assumed that $\Omega$ is moderately small such that the term $(\rho'\Omega^2/2)\boldsymbol{\nabla}\left|\hat{z}\times\boldsymbol{r}\right|^2$ can be neglected. It is critically important to notice that the terms $\boldsymbol{g}_{static}\rho'$ and $\boldsymbol{g}'\rho_{static}$ in (4) are generally of the same order of magnitude. This is because

$$|\boldsymbol{g}_{static}\rho'| \sim \left|\rho'\boldsymbol{\nabla}\left[\int_0^\pi\int_0^{\widetilde{R}(\tilde{\theta})}\frac{\tilde{r}^2\sin\tilde{\theta}\rho_{static}}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}\right]\right| = \mathrm{O}\left(\rho'\rho_{static}\right); \quad (6)$$
$$|\boldsymbol{g}'\rho_{static}| \sim \left|\rho_{static}\boldsymbol{\nabla}\left[\int_0^\pi\int_0^{\widetilde{R}(\tilde{\theta})}\frac{\tilde{r}^2\sin\tilde{\theta}\rho'}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}\right]\right| = \mathrm{O}\left(\rho'\rho_{static}\right). \quad (7)$$

Physically, it simply means that, when the internal density anomaly $\rho'$ is induced by the deep flow $\boldsymbol{u}$, the hydrostatic gravitational force $\boldsymbol{g}_{static}$ must be also perturbed to yield the concomitant gravitational perturbation $\boldsymbol{g}'$. Upon making a spherical approximation, we obtain

$$2\Omega\int_{\pi/2}^{\theta}\left[\cos\tilde{\theta}\frac{\partial}{\partial r}-\frac{\sin\tilde{\theta}}{r}\frac{\partial}{\partial\tilde{\theta}}\right](\rho_{static}U)\mathrm{d}\tilde{\theta} = \frac{g_{static}(r)}{r}\rho'(r,\theta)-\frac{2\pi Gq(r)}{r}\times\int_0^\pi\int_0^{R_s}\frac{\tilde{r}^2\rho'(\tilde{r},\tilde{\theta})}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}\sin\tilde{\theta}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}+C(r), \quad (8)$$

where $\boldsymbol{r} = \boldsymbol{r}(r,\theta)$, $\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}(\tilde{r},\tilde{\theta})$ and $C(r)$ is an arbitrary function of $r$. Equation (8) represents a two-dimensional kernel integral equation which is referred to as *the thermal-gravitational wind equation* (TGWE). The two-dimensional kernel integral TGWE (8) that contains the Green's function $1/|\boldsymbol{r}-\tilde{\boldsymbol{r}}|$ in its integrand can be solved by an extended spectral method [7].

## 3. The TWE vs. the TGWE

We demonstrate that the solution of the TGWE (8) differs substantially from that of the TWE (1) for exactly the same model and parameter values and, hence, the TWE cannot generally provide a reasonable approximation to the TGWE. For this purpose, we introduce three characteristic quantities for measuring the difference between the TWE and TGWE solutions First, we introduce the norm $\Delta_{\mathrm{diff}}$ defined as

$$\Delta_{\mathrm{diff}} = \frac{||\rho'_{\mathrm{TGWE}}(\boldsymbol{r})-\rho'_{\mathrm{TWE}}(\boldsymbol{r})||_2}{||\rho'_{\mathrm{TWE}}(\boldsymbol{r})||_2},$$

where the solution $\rho'$ of the TWE (1) is denoted as $\rho'_{\mathrm{TWE}}$, the solution $\rho'$ of the TGWE (8) as $\rho'_{\mathrm{TGWE}}$ and

$$||F||_2 = \left[\int_0^{2\pi}\int_0^\pi\int_0^{R_s}|F(\boldsymbol{r})|^2\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\right]^{1/2},$$

to measure the difference between $\rho'_{\mathrm{TGWE}}$ and $\rho'_{\mathrm{TWE}}$. The second characteristic quantity, adopted in the case of an equatorially antisymmetric wind, is the distance $\Delta z$ between the center of mass and the center of figure caused by the wind-induced density anomaly. The third characteristic quantity, adopted in the case of an equatorially symmetric wind, is the lowermost coefficient $(J_2)_{\mathrm{TGWE}}$ computed from $\rho'_{\mathrm{TGWE}}$ and $(J_2)_{\mathrm{TWE}}$ from $\rho'_{\mathrm{TWE}}$. We found, for a typical Jupiter-like model with the deep zonal winds, that

$$\frac{||\rho'_{\rm TGWE}(\boldsymbol{r}) - \rho'_{\rm TWE}(\boldsymbol{r})||_2}{||\rho'_{\rm TWE}(\boldsymbol{r})||_2} = {\rm O}(100\%),$$

$$\frac{[(\Delta z)_{\rm TGWE} - (\Delta z)_{\rm TWE}]}{(\Delta z)_{\rm TWE}} = {\rm O}(100\%),$$

and

$$\frac{[(J_2)_{\rm TGWE} - (J_2)_{\rm TWE}]}{(J_2)_{\rm TWE}} = {\rm O}(100\%).$$

It reconfirms the result of the order-of-magnitude analysis: the gravitational perturbation term in the TGWE (8) neglected in the TWE (1) generally makes a leading-order contribution and, hence, must be retained.

## 4. Summary and Conclusion

The present study shows that the TWE (1) is generally incorrect for the purpose of computing the gravitational signature of a giant gaseous planet caused by the zonal winds in its deep interior. This is because an extra term representing the concomitant gravitational perturbation $\boldsymbol{g}'$ produced by the density anomaly $\rho'$ is of the same order of magnitude and, hence, must be retained, leading to the TGWE (8).

There exist, however, two special circumstances in which the kernel integral in the TGWE (8) can be neglected and, consequently, the TWE (1) provides a good approximation to the TGWE (8). The first circumstance is when the interior fluid of a planet is weakly compressible everywhere, *i.e.*,

$$\left| \frac{1}{\rho_{static}(r)} \frac{{\rm d}\rho_{static}(r)}{{\rm d}r} \right| \ll 1 \ \text{ in } \ 0 < r < R_s.$$

However, this case does not represent the typical interior of a giant gaseous planet like Jupiter which is believed to be strongly compressible. The second circumstance is when the zonal winds $U$ and the corresponding wind-induced density perturbation $\rho'$ are primarily confined within a very thin outer layer defined by $(R_s - \epsilon) \leq r \leq R_s$ with $0 < (\epsilon/R_s) \ll 1$. However, this case represents an uninteresting trivial case not only because we are concerned with how a deep wind induces an externally measurable gravitational signature but also because it is obvious that $\Delta J_n^{dyn} \to 0$ when $(\epsilon/R_s) \to 0$.

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