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Group Theoretical Approach to Polarisation in Radiative Transfer Theory

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Vectorial Radiative Transfer (VRT) models take the polarisation of light into account. The polarisation state of light is usually encoded by Stokes' vector. However, based on Group Theory, it is found that one can represent electromagnetic polarisation in six different ways.

At any point in space, the polarisation state of a plane harmonic electromagnetic wave is already implicitly encoded in the representative of its transversal electric field $\mathbf{E}$. The complex representative of $\mathbf{E}$ is an element of a 2-dimensional complex space, which we denote by $\mathcal{E}$. What we usually want is an alternative more explicit encoding of the polarisation state, which better reveals the geometry of the state of polarisation and its parameters. Denote by $\mathcal{P}$ the space of these more explicit polarisation states.

Any change in $\mathbf{E}$, along the line of propagation, can be represented by a transformation in $\mathcal{E}$ and is necessarily an element of the group $GL\,(2,\mathbb{C})$, while a pure change of polarisation state is represented by an element of the subgroup $SL\,(2,\mathbb{C})$. To any change in polarisation of $\mathbf{E}$ must correspond a transformation in $\mathcal{P}$, represented by an element of some group, say G. The map $\Gamma : SL\,(2,\mathbb{C}) \to G$ must be a group preserving homomorphism.

Based on Group Theory, we have the following cases.

(A) The map Γ is an isomorphism.

(A.1) $G = Spin_+\,(1,3)$ and $\mathcal{P}$ is 2-dimensional quaternion space $\mathbb{H}^2$.

(A.2) $G = Spin_+\,(3,1)$ and $\mathcal{P}$ is 4-dimensional real space $\mathbb{R}^4$.

(A.3) $G = SV(2)$, the Special Vahlen group in 2 real dimensions, and $\mathcal{P}$ is 2-dimensional hyperbolic quaternion space.

(A.4) $G = Sp(2,\mathbb{C})$ the group of complex symplectic 2×2 matrices, and $\mathcal{P}$ is $\mathbb{C}^2$.

(A.5) $G = SL(2,C)$, the group of complex 2×2 matrices with unit determinant, and $\mathcal{P}$ is $\mathbb{C}^2$ (Jones' calculus).

(A.6) $G = Spin(3,\mathbb{C})$ the spinor group of 3-dimensional complex space, and $\mathcal{P}$ is $\mathbb{C}^3$.

Case A gives rise to five alternative calculi, besides the Jones calculus, for fully polarised light.

(B) The map Γ is a $2-1$ homomorphism.

(B.1) $G = SO_+(1,3)$, the restricted Lorentz group for signature $(1,3)$, and $\mathcal{P}$ is $R^{1,3}$ (Mueller's calculus);

(B.2) $G = SO_+(3,1)$, the restricted Lorentz group for signature $(3,1)$, and $\mathcal{P}$ is $R^{3,1}$;

(B.3) $G = SM\ddot{o}b\,(2)$, the Special Möbius group in 2 real dimensions, and $\mathcal{P}$ is the extended complex plane $\widehat{\mathbb{C}}$;

(B.4) $G = CS^2$, the Conformal group of S^2, and $\mathcal{P}$ is the Riemann sphere S^2;

(B.5) $G = PSL(2,\mathbb{C})$, the Projective Special Linear group in 2 complex dimensions, and $\mathcal{P}$ is the complex projective line CP^1;

(B.6) $G = SO(3,\mathbb{C})$, the Special Orthogonal group in 3 complex dimensions, and $\mathcal{P}$ is 3-dimensional complex space $\mathbb{C}^3$.

Case B gives rise to five additional ways, besides Stokes' choice (B.1), to encode polarisation.

This implies that also six different formulations of VRT theory can be given. In addition, it reveals what the structure of the extinction matrix is: namely an element of the common Lie algebra of the above Lie groups G. Finally, this shows that analytically solving the VRT problem, which is theoretically possible in the absence of multiple scattering, will involve an element of G, depending on the choice made to represent polarisation.

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